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3.2 Force of mortality (dødsintensitet)

force of mortality of (x) at x+t :

$$\mu_{x+t} := \frac{g(t)}{1-G(t)} = -\frac{d}{dt}\log(1-G(t))$$

$\longrightarrow$ measure for the speed of mortality

Since

$$g(t) = \underbrace{{}_tp_x}_{1-G(t)} \cdot \mu_{x+t}$$

(3.1.5) implies that

$$\overset{\circ}{e}_x = \int_0^\infty t \; {}_tp_x \, \mu_{x+t} \, dt \tag{3.2.1}$$

Further, since

$$\mu_{x+t} = -\frac{d}{dt}\log {}_tp_x$$

we get by integration that

$${}_tp_x = e^{-\int_0^t \mu_{x+s}\,ds} \tag{3.2.2}$$

Rem. 3.2.1 : Because of (3.1.4) we observe that

$${}_sq_{x+t} \approx \mu_{x+t} \cdot s \quad \text{for small } s$$

Now let us discuss some explicit distributions for T

3.3 Analytical distributions of T

$\longrightarrow$ explicit formula for G

$\longrightarrow$ favourable from the computational point of view

Examples :

1. de Moivre (1724) :

$$g(t) = \frac{1}{\omega - x}, \; 0 < t < \omega - x \quad \text{(uniform distr. on } [0, \omega-x])$$

ω maximum age

$$\Longrightarrow \mu_{x+t} \overset{\text{def}}{=} \frac{g(t)}{1-\underbrace{G(t)}_{\frac{t}{\omega-x}}} = \frac{1}{\omega-x-t}, \; 0 < t < \omega - x$$

2. Gompertz (1824) :

$$\mu_{x+t} := B\,c^{x+t}, \; t > 0$$

3. Makeham (1860) :

$$\mu_{x+t} := \underbrace{A}_{\substack{\geq 0 \\ \text{indep. of age}}} + B\,c^{x+t}, \; t > 0$$

$\overset{(3.2.2)}{\Longrightarrow}$

$${}_tp_x = \exp(-At - m\,c^x(c^t - 1)), \; m = B/\log c$$

4. Weibull (1939) :

$$\mu_{x+t} = \underset{\geq 0}{k}\,(x+t)^{n>0}$$

! Exerc. : Calc. the survival prob. !

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For later use we have to introduce some r.v.'s related to T.

3.4 Curtate future lifetime of (x)

1. K curtate future lifetime of (x):

$$K := [T]$$ Gauss bracket; rounds off to the next integer

e.g. $[1.7] = 1$, $[2.9] = 2$, $[3.1] = 3$

$\longrightarrow$ # of completed future years lived by (x)

$\overset{\text{def.}}{\Longrightarrow}$ $\Pr(K = j) = \Pr(j \leq T < j+1) = {}_j p_x \cdot {}_1 q_{x+j}$, $\quad$ (3.4.1)

$j \in \mathbb{N}_0$

surv. prob.

cond. prob. to die within 1 year, after having reached $x+j$ years

expected value of K:

$$e_x := E[K] = \sum_{j \geq 1} j \Pr(K = j) \overset{(3.4.1)}{=} \sum_{j \geq 1} j \cdot {}_j p_x \cdot {}_1 q_{x+j} \qquad (3.4.2)$$

or $\quad e_x = \sum_{j \geq 1} \Pr(K \geq j) = \sum_{j \geq 1} {}_j p_x$

2. S: $\quad S := T - K \in [0, 1)$

$\rightarrow$ remaining lifetime in the year of death

Rem. 3.4.1: (i) S has cont. distr. on $[0, 1]$

(ii) $\mathring{e}_x \approx e_x + \frac{1}{2}$

3. $S^{(m)}$:

$$S^{(m)} := \frac{1}{m} [m \cdot S + 1]$$ Gauss bracket

rounds to the next higher multiple of $\frac{1}{m}$

e.g. $S = \frac{2}{m} + \frac{1}{2m} \rightarrow S^{(m)} = \frac{3}{m}$

Next we want to give an approximation of the distribution of T by means of one-year survival probabilites

3.5 Approximation of $G(t)$

Consider 1-year survival prob.

p_{x+j}, $j = 0, 1, 2, \ldots$

given by a <u>life table</u> (see later)

$\longrightarrow$ ${}_k p_x$, $k \in \mathbb{N}$ known, since

$${}_k p_x = p_x \, p_{x+1} \, p_{x+2} \cdots p_{x+k-1}, \quad k \in \mathbb{N}$$

$\longrightarrow$ <u>objective</u>: approximation of $G(t) = \Pr(T \leq t)$

$\longrightarrow$ method: <u>interpolation</u> of ${}_u q_x$ or μ_{x+u} at intermediate ages $x+u$, $x \in \mathbb{N}_0$, $0 < u < 1$

(15) e.g. if μ_{x+u} known by interpolation

$\Longrightarrow \quad {}_tp_x \overset{(3.2.2)}{=} e^{-\int_0^t \mu_{x+s}\,ds}$ known for all t

$\Longrightarrow \quad G(t) = 1 - {}_tp_x$ known for all t

1. method :

assumption : (i.e. ${}_uq_{x+j} \overset{u.}{=} u\, q_{x+j}$, $j \in \mathbb{N}_0$)

$${}_uq_x = u \cdot q_x \;,\; 0 \le u \le 1 \qquad (\text{linearity of } {}_uq_x)$$

$\Longrightarrow \quad {}_up_x = 1 - {}_uq_x$

$\overset{\text{def.}}{\Longrightarrow} \quad \mu_{x+u} = \dfrac{q_x}{1 - u \cdot q_x} \qquad (3.5.1)$

Rem. 3.5.1 : (3.5.1) holds, if e.g. S uniformly distributed and K, S are indep. (see Exc. 2)

2. method :

assumption : $\mu_{x+u} = \mu_{x+\frac{1}{2}} \;,\; 0 < u < 1$

(16) We now consider stochastic payment streams depending on the future lifetime T

4. Life insurance

An insurance policy is specified by

1. benefits payable by the insurer
2. premiums payable by the insured

Assumption: benefits and premiums may depend on the r.v. T

$Z :=$ PV of the benefits

$V :=$ PV of the premium payments

$\Rightarrow$ Z, V r.v.'s

$\longrightarrow$ $L := Z - V$ total loss of the insurer

$\longrightarrow$ reasonable to require that

$$E[L] = 0$$

$$\Leftrightarrow \quad E[V] = E[Z]$$

$\longrightarrow$ $E[Z]$ net single premium

Remark 4.1: $E[Z]$ is not a measure for the risk of the insurer

$\longrightarrow$ further characteristics of the distribution of Z (e.g. tails of distributions) are required for the assessment of risk

Examples of insurance policies based on T:

4.1 Elementary insurance types

4.1.1 Whole life and term insurance

1. whole life insurance (Enkel livsforsikring):

$$Z := v^{K+1} \quad \text{at } t = K+1$$

$$K = [T]$$

$$v = \frac{1}{1+i}$$

$\longrightarrow$ payment 1 at the end of the year of death

(17) $\longrightarrow$ net single premium:

$$A_x := E[Z] = E[v^{K+1}]$$

$$= \sum_{j\geq 0} v^{j+1} Pr(K=j)$$

$$\overset{(3.4.1)}{=} \sum_{j\geq 0} v^{j+1}\, {}_jp_x\, q_{x+j}$$

$\longrightarrow$ "risk measure" $Var[Z]$:

$$Var[Z] = E[Z^2] - A_x^2$$

$v = e^{-\delta}$, δ force of interest

$$\Rightarrow E[Z^2] = E[e^{-2\delta(K+1)}] = \sum_{j\geq 0} e^{-2\delta(j+1)}\, {}_jp_x\, q_{x+j}$$

2. term insurance: (risiko- eller korttidsforsikring)

$$Z = v^{K+1}\cdot 1_{\{K<n\}} \quad \text{at } t = K+1$$

$\longrightarrow$ payment of 1, if year of death $< n$ duration of the contract otherwise zero

$\longrightarrow$ net sing. prem:

$$A^1_{x:\overline{n|}} := E[Z] = \sum_{j=0}^{n-1} v^{j+1}\, {}_jp_x\, q_{x+j}$$

4.1.2 Pure endowments

pure endowment of duration n:

$$Z = v^n\cdot 1_{\{K\geq n\}} \quad \text{at } t = n$$

$\longrightarrow$ paym. of 1, if the insured is alive at the end of n years else zero

$\longrightarrow$ net sing. prem.:

$$A_{x:\overline{n|}}^{\;\;1} := E[Z] = v^n\, {}_np_x$$

4.1.3 Endowments (sammensatt livsforsikring)

$\longrightarrow$ mixed case

$\longrightarrow$ Ex. 4.1.3.1:

$$Z = \underbrace{\overbrace{v^{K+1} 1_{\{K<n\}}}^{=:Z_1}}_{\text{term insur.}} + \underbrace{\overbrace{v^n\cdot 1_{\{K\geq n\}}}^{=:Z_2}}_{\text{pure endowment}} \quad \text{at } t = K+1,\, n$$

$\longrightarrow$ net single prem.:

$$A_{x:\overline{n|}} := E[Z] = E[Z_1] + E[Z_2]$$

$$= A^1_{x:\overline{n|}} + A_{x:\overline{n|}}^{\;\;1}$$

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4.2 Payment at the moment of death

Consider e.g.

$$Z = v^T \qquad \text{at } t = T$$

$\longrightarrow$ paym. of 1 at $t=T$, i.e. at the instant of death

$\longrightarrow$ net sing. prem.:

$$\bar{A}_x \overset{!}{=} E[Z] = \int_0^\infty v^t \underbrace{g(t)}_{= {}_t p_x \, \mu_{x+t}} dt$$

4.3 General types of life insurance

We consider benefits depending on time:

1. c_j payments at j-th year:

$$Z = c_{K+1} v^{K+1}$$

$$\Rightarrow E[Z] = \sum_{j \geq 0} c_{j+1} v^{j+1} {}_j p_x \cdot q_{x+j}$$

2. $c(t)$ cont. paym.:

$$Z = c(T) v^T$$

$$\Rightarrow E[Z] = \sum_{j \geq 0} E[Z \mid K=j] \cdot \Pr(K=j)$$

$$\overset{T=K+S}{=} \sum_{j \geq 0} E[c(j+S) v^{j+S} \mid K=j] \cdot \underbrace{\Pr(K=j)}_{= {}_j p_x \cdot q_{x+j}} \qquad \text{(see Exerc. 2, Prob. 8)}$$

4.4 Equations for A_x, $\bar{A}_x$

1. A_x: $\quad$ net sing. pr. of a whole life ins.

$$A_x = E[v^{K+1}] = \sum_{j \geq 0} v^{j+1} {}_j p_x \, q_{x+j}$$

$${}_j p_x = p_x \cdot {}_{j-1} p_{x+1}$$

$$\Rightarrow A_x = v^{0+1} \underbrace{{}_0 p_x}_{=1} \cdot q_{x+0} + \sum_{j \geq 1} \underbrace{v^{j+1} {}_j p_x \, q_{x+j}}_{= v^{j+1} p_x \cdot {}_{j-1} p_x \cdot q_{x+j}}$$

$$= v \cdot q_x + v \cdot p_x \underbrace{\sum_{j \geq 0} v^j {}_j p_{x+1} \cdot q_{x+1+j}}_{= A_{x+1}} \qquad \left(v^{j+1} = v \, v^{j-1}, \quad q_{x+j} = q_{(x+1)+(j-1)}\right)$$

$$\Rightarrow A_x = v q_x + v A_{x+1} p_x \qquad (4.4.1)$$

$$= E[Y],$$

where

$$Y = \begin{cases} v, & \text{if } T < 1 \\ v A_{x+1}, & T \geq 1 \end{cases}$$

$\xrightarrow{(4.4.1)}$

$$A_x = v A_{x+1} + v(1 - A_{x+1}) q_x = \underbrace{v A_{x+1}}_{\text{reserve}} + \underbrace{A^1_{x:\overline{1}|} \cdot (1 - A_{x+1})}_{\substack{\text{net sing. prem. of a} \\ \text{1-year term ins. with} \\ \text{paym. } (1-A_{x+1})}}$$

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2. $\bar{A}_x$: $\bar{A}_x = E[v^T] = \underbrace{E[v^T \mid T \le h] \cdot {}_h q_x + v^h \overbrace{E[v^{T-h} \mid T > h]}^{= \bar{A}_{x+h}} \cdot {}_h p_x}$

net sing. prem. of the same person aged x+h

$$\Rightarrow \frac{\bar{A}_{x+h} - \bar{A}_x}{h} = \underbrace{\frac{(1 - v^h \cdot {}_h p_x)}{h}}_{\xrightarrow{h \to 0} \delta + \mu_x} \bar{A}_{x+h} - \underbrace{\frac{E[v^T \mid T \le h] \cdot {}_h q_x}{h}}_{\xrightarrow{h \to 0} g(0) = \mu_x}$$

$$\xRightarrow{h \to 0} \quad \delta \cdot \bar{A}_x = \frac{d}{dx} \bar{A}_x + \mu_x (1 - \bar{A}_x) \qquad (4.4.2)$$

5. Life annuities

life annuities $\longrightarrow$ e.g. annuity-due, immediate annuity depending on the future life time T

Examples $\longrightarrow$

5.1 Elementary life annuities

Ex 5.1.1 : whole life annuity-due (straks begynnende annuitet av varighet K+1):

$$Y = \sum_{j=0}^{K} v^j = \ddot{a}_{\overline{K+1}|}$$

$\longrightarrow$ PV of annual paym. of 1 until n=K (starting in zero)

$\longrightarrow$ net sing. prem.:

$$\ddot{a}_x := E[Y] = \sum_{j \ge 0} \ddot{a}_{\overline{j+1}|} \underbrace{Pr(K=j)}_{= {}_j p_x \cdot q_{x+j}}$$

Rewrite Y as

$$Y = \sum_{j \ge 0} \underbrace{v^j \cdot 1_{\{K \ge j\}}}_{\text{PV of a pure endowment}}$$

$$\xRightarrow{E[\cdot]} \quad \ddot{a}_x = \sum_{j \ge 0} A_{x:\overset{1}{\overline{j}|}}$$

Ex. 5.1.2 : n-year temporary life annuity-due

$$Y = \ddot{a}_{\overline{K+1}|} \cdot 1_{\{K<n\}} + \ddot{a}_{\overline{n}|} \cdot 1_{\{K \ge n\}}$$

$\longrightarrow$ PV of an whole life annuity-due, if K<n duration or of an annuity-due, if K≥n

$\longrightarrow$ net sing. prem.

$$\ddot{a}_{x:\overline{n}|} := E[Y] = \sum_{j=0}^{n-1} \ddot{a}_{\overline{j+1}|} \, {}_j p_x \, q_{x+j} + \ddot{a}_{\overline{n}|} \cdot {}_n p_x$$

Ex. 5.1.3 : immediate life annuity:

$$Y = \sum_{j=1}^{K} v^j = a_{\overline{K}|}$$

$\longrightarrow$ PV of paym. of 1 from t=1 to t=K

5.2 Life annuity with m payments per year

$$Y = \ddot{a}^{(m)}_{\overline{K+S^{(m)}|}}$$

$\longrightarrow$ PV of paym. $\frac{1}{m}$ m times per year from $t=0$ to $t = K + S^{(m)} - \frac{1}{m}$; $S^{(m)} = \frac{1}{m}[mS+1]$ m-th fraction of the year in which the insured dies

$\longrightarrow$ net sing. prem.

$$\ddot{a}^{(m)}_x := E[Y]$$

Rem. 5.2.1: If K, S indep., S unif. distr. on $[0,1]$ then

$$\ddot{a}^{(m)}_x = \alpha(m)\ddot{a}_x - \beta(m),$$

where $\alpha(m) \approx 1$ and $\beta(m) \approx \frac{m-1}{2m}$ (see Exerc. 2)

5.3 Variable life annuities

$$Y = \sum_{j=0}^{K} r_j v^j = \sum_{j\geq 0} r_j v^j 1_{\{K \geq j\}}$$

$\longrightarrow$ PV of paym. of $r_1, \dots, r_K$

$\longrightarrow$ net sing. prem.

$$E[Y] = \sum_{j \geq 0} r_j v^j {}_jp_x$$

Consider the continuous-time case:

$r(t), t \geq 0$ paym. rate at time t

$$\longrightarrow \quad Y = \int_0^T v^t r(t)\, dt$$

$\longrightarrow$ net sing. prem.

$$E[Y] = \int_0^\infty v^t r(t) {}_tp_x \, dt \qquad \text{(see Exerc. 3)}$$

5.4 Equations for $\ddot{a}_x, \bar{a}_x$

1. $\ddot{a}_x$: From

$$\ddot{a}_x = \sum_{j\geq 0} v^j {}_jp_x$$

and

$$ {}_jp_x = p_x \cdot {}_{j-1}p_{x+1}$$

it follows

$$\ddot{a}_x = 1 + v \ddot{a}_{x+1} \cdot p_x \qquad (5.4.1)$$

$$\Longleftrightarrow \quad \ddot{a}_x = \underbrace{1 + v\ddot{a}_{x+1}}_{\text{bond}} - \underbrace{v\ddot{a}_{x+1} q_x}_{\text{expected mortality gain}}$$