(21) 2. $\bar{a}_x$! $\bar{a}_x := \lim_{m\to\infty} \ddot{a}_x^{(m)}$

One shows that

$$1 = \underbrace{m\left(1-(1+i)^{-1/m}\right)}_{\xrightarrow[m\to\infty]{} \delta} \ddot{a}_x^{(m)} + \underbrace{A_x^{(m)}}_{\xrightarrow[m\to\infty]{} \bar{A}_x = E[v^T]} \qquad (5.4.2)$$

$A_x^{(m)} := E[v^{K+S^{(m)}}]$, $S^{(m)} \to S$ for $m\to\infty$

$$\Rightarrow \quad 1 = \delta \bar{a}_x + \bar{A}_x \qquad (5.4.3)$$

$$\xRightarrow{(4.4.2)} \quad \delta \bar{a}_x = 1 + \frac{d}{dx}\bar{a}_x - \mu_x \bar{a}_x$$

5.5 Calculation of $\ddot{a}_{x+u}$, $x \in \mathbb{N}_0$, $0<u<1$

Observe that

$$_u p_x \cdot {_k p_{x+u}} = {_k p_x}\, {_u p_{x+k}}$$

Ass. (see Sect 3.5) : $_u q_{x+j} = u \cdot q_{x+j}$, $0<u<1$

$$\Rightarrow \quad (1-u\cdot q_x)\, {_k p_{x+u}} = {_k p_x} \cdot (1-u\cdot q_{x+k})$$

Multiplication with v^k and summation over k give

$$(1-u\cdot q_x)\cdot \ddot{a}_{x+u} = \ddot{a}_x - u(1+i)A_x$$

← recursion for $\ddot{a}_x$

Then, using (5.4.2) for $m=1$ and (5.4.1) we get

$$\ddot{a}_{x+u} = \frac{1-u}{1-u\cdot q_x}\ddot{a}_x + \frac{u(1-q_x)}{1-u\cdot q_x}\ddot{a}_{x+1}$$

$$\Rightarrow \quad \ddot{a}_{x+u} \approx (1-u)\ddot{a}_x + u\cdot \ddot{a}_{x+1},$$

if q_x small

22

6. Net premiums

Recall that the total loss L of the insurer
; or the PV of insurer's liability
is given by

$$L = Z - V.$$

Z := PV of the benefits payable by the insurer
V := PV of the premiums provided by the insured

$\longrightarrow$ reasonable requirement !

$$E[L] = 0 \qquad (6.1)$$

$\longrightarrow$ condition (6.1) is called
<u>equivalence principle</u>

<u>Forms of premium payments</u> :

1. 1 single prem.
2. constant periodic premiums
3. variable periodic premiums

In the sequel we <u>confine</u> ourselves to the 2 cases 1 or 2

<u>Def. 6.1</u> (net premium)
A premium satisfying the equivalence principle (6.1) is called <u>net premium</u>.

<u>Ex. 6.2</u> (term insurance)

$$Z = C \cdot v^{K+1} 1_{\{K<n\}}$$

$\longrightarrow$ PV of a paym. C (sum insured) at $t = K+1$, if $K < n$ duration

$$V = \Pi \cdot \ddot{a}_{\overline{K+1|}} 1_{\{K<n\}} + \Pi \ddot{a}_{\overline{n|}} 1_{\{K \geq n\}}$$

$\longrightarrow$ PV of annual paym. Π (premium) until $\min(K, n-1)$

$\Longrightarrow$ total loss

$$L = \left(C \cdot v^{K+1} - \Pi \ddot{a}_{\overline{K+1|}}\right) 1_{\{K<n\}} - \Pi \ddot{a}_{\overline{n|}} 1_{\{K \geq n\}}$$

23

$$E[L] = 0$$

$$\Longleftrightarrow \quad E[Z] = E[V]$$

$$\Longleftrightarrow \quad C \underbrace{A^{1}_{x:\overline{n|}}}_{\text{net sing. prem. of term insur.}} = \Pi \underbrace{\ddot{a}_{x:\overline{n|}}}_{\text{net sing. prem of an year temp. life ann.}}$$

$$\Longrightarrow \quad P^{1}_{x:\overline{n|}} = \Pi = \frac{C A^{1}_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}} \qquad (6.2)$$

<u>deficiency</u> of net premiums based on the equivalence principle:
Cond. (6.1) <u>does not</u> reflect the insurer's risk or risk preferences

$\longrightarrow$ <u>Alternative condition to (6.1)</u> to capture risk more appropriately:
utility-based equivalence principle:

$$E[u(-L)] = u(0), \qquad (6.3)$$

where
$u(x)$ <u>utility function</u>, i.e. $u'(x) > 0$ and $u''(x) < 0$

$\longrightarrow$ $u(x)$ measures the utility of the amount of money x from the view point of the insurer

<u>Interpretation</u> of (6.3):
The expected utility of the insurer's "gain" (i.e. $x = -L$) should be equal to the utility of the "gain" zero

$\longrightarrow$ (6.3) "provides a "fairness condition" to <u>calculate net premiums</u>"

$\longrightarrow$ <u>Ex. 6.3</u>: $u(x) = \frac{1}{a}(1 - e^{-ax})$

$a > 0$ <u>risk aversion</u> of the insurer (grad av uvilje å ta risiko)

Choose L as in Ex. 6.2.

$\Longrightarrow$ (6.3) $\Longleftrightarrow$ $E[e^{aL}] = 1$

$$\Longleftrightarrow \sum_{j=0}^{n-1} \exp(a \cdot C v^{j+1} - a \Pi \ddot{a}_{\overline{j+1|}}) \, {}_{j}p_x \cdot q_{x+j} + \exp(-a \Pi \ddot{a}_{\overline{n|}}) \cdot {}_{n}p_x = 1$$

(24) | solvable by interval bisection or Newton-Raphson method

Rem. 6.4 :

Π_{eq} premium based on (6.1)
and Π_u prem. calculated from (6.3)

$\longrightarrow$ $\Pi_u > \Pi_{eq}$

$\longrightarrow$ $R := \Pi_u - \Pi_{eq}$ safety loading to cover the insurer's risk

Further examples of premium calculation w.r.t. (6.1) :

Ex. 6.5 (whole life insur.)

$Z = v^{K+1}$

$\longrightarrow$ PV of a paym. 1 at $t = K+1$

$V = \Pi \cdot \ddot{a}_{\overline{K+1|}}$

$\longrightarrow$ PV of ann. paym. of Π until K (starting in zero)

$\Longrightarrow$ $L = v^{K+1} - \Pi \cdot \ddot{a}_{\overline{K+1|}}$

$\longrightarrow$ net premium :

$$P_x := \frac{E[v^{K+1}]}{E[\ddot{a}_{\overline{K+1|}}]} = \frac{A_x}{\ddot{a}_x}$$ $\leftarrow$ net sing. prem. of whole life ann.

$\ddot{a}_{\overline{K+1|}} = \frac{1-v^{K+1}}{d}$, $d \overset{\text{def}}{=} \frac{i}{1+i}$ discount rate

$\Longrightarrow L = \left(1+\frac{P_x}{d}\right)v^{K+1} - \frac{P_x}{d} \Longrightarrow$

$$\underbrace{Var[L]}_{\text{risk with net annual premiums}} = \left(1+\frac{P_x}{d}\right)^2 Var[v^{K+1}] > \underbrace{Var[v^{K+1}]}_{\text{risk with net single premium } A_x} = Var[v^{K+1} - A_x]$$

Ex. 6.6 (pure endowment)

$Z = v^n 1_{\{K \geq n\}}$

$\longrightarrow$ PV of a paym. of 1 at $t = n$, if $K \geq n$, else zero

$V = \Pi \cdot \ddot{a}_{\overline{K+1|}} 1_{\{K<n\}} + \Pi \cdot \ddot{a}_{\overline{n|}} 1_{\{K \geq n\}}$

$\longrightarrow$ PV of annual premiums Π until $t = \min(K, n-1)$

$\longrightarrow$ net prem. :

$$P_{x:\overset{1}{\overline{n|}}} := \Pi = \frac{A_{x:\overset{1}{\overline{n|}}}}{\ddot{a}_{x:\overline{n|}}}$$

$\leftarrow$ net sing. prem. of a pure endowment

$\leftarrow$ net sing. prem. of an n-year temp. life ann.-due

Ex. 6.7 (endowment)

$$P_{x:\overline{n|}} := \Pi = \frac{A_{x:\overline{n|}}}{\ddot{a}_{x:\overline{n|}}}$$ $\leftarrow$ net sing. prem. of an endowment

7. Net premium reserves

7.1 Motivation

In the following denote by

$_tZ$ PV of the benefits at future time $t \geq 0$

and $_tP$ PV " premiums "

Then

$$_tL \stackrel{\text{def}}{=} {}_tZ - {}_tP \qquad (7.1.1)$$

is called the total <u>loss of the insurer at future time $t \geq 0$</u> $\quad ({}_0L = L)$

$\longrightarrow$ fundamental concept of life insurance

<u>Def. 7.1.1</u> (net premium reserves)

The cond. expect.

$$_tV \stackrel{\text{def}}{=} E[{}_tL \mid T > t]$$

is called <u>net premium reserve at time $t \geq 0$</u>

<u>In order to keep</u> the insured <u>interested</u> in continuing the insurance it is reasonable to assume that

$$_tV \geq 0 \quad \text{for all } t \geq 0$$

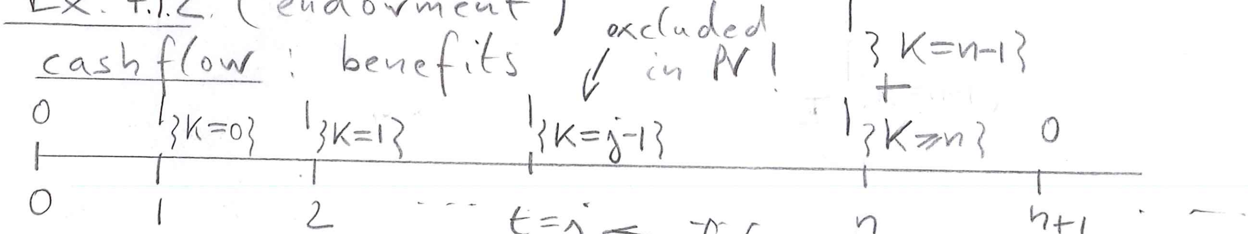
<u>On the other hand</u> it is plausible that the "average" total loss V_t could be sufficient to <u>compensate for the insurer's liability</u> $_tL$ at time t

$\longrightarrow$ reasonable to require the following <u>"fairness" condition</u>:

1. $E[L] = 0$ (equivalence principle)
2. $_tV \geq 0, \; t \geq 0 \qquad (7.1.2)$

<u>Ex. 7.1.2.</u> (endowment)

<u>cash flow</u>: benefits



$$\longrightarrow \quad {}_jZ = \sum_{l=j+1}^{n} v^{l-j} \cdot 1_{\{K=l-1\}} + v^{n-j} 1_{\{K \geq n\}}$$

$$\Longrightarrow \quad E[{}_jZ \mid T > j] = \sum_{l=j+1}^{n} v^{l-j} \underbrace{E[1_{\{K=l-1\}} \mid T > j]}_{= Pr(K=l-1 \mid T>j) = {}_{l-1-j}p_{x+j} \cdot q_{x+l-1}}$$

$$+ \, v^{n-j} \underbrace{E[1_{\{K \geq n\}} \mid T > j]}_{= {}_{n-j}p_{x+j}}$$

$$= \sum_{K=0}^{n-j-1} v^{K+1} \, {}_Kp_{x+j} \cdot q_{x+j+K} + v^{n-j} \, {}_{n-j}p_{x+j} = A_{x+j:\overline{n-j|}}$$

net sing. prem. of an endowment

cash flow: premiums

$$\Pi = P_{x:\overline{n|}} \quad \Pi\cdot 1_{\{K\geq 1\}} \quad \Pi\cdot 1_{\{K\geq 2\}} \quad \ldots \quad \Pi\cdot 1_{\{K\geq j\}} \quad \ldots \quad \Pi\cdot 1_{\{K\geq n-1\}} \quad 0$$

$$0 \quad 1 \quad 2 \quad \ldots \quad t=j \quad \ldots \quad n-1 \quad n \quad \ldots$$

$t=j \leftarrow$ future time

$$\longrightarrow \quad {}_jP = \sum_{l=j}^{n-1} v^{l-j}\cdot\Pi\cdot 1_{\{K\geq l\}}$$

$$\Longrightarrow \quad E[{}_jP \mid T>j] = \sum_{l=j}^{n-1} v^{l-j}\cdot\Pi\cdot \underbrace{E[1_{\{K\geq l\}} \mid T>j]}_{=\Pr(K\geq l \mid T>j) = {}_{l-j}p_{x+j}}$$

$$= \sum_{l=j}^{n-1} v^{l-j}\,\Pi\,{}_{l-j}p_{x+j} = \sum_{\mu=0}^{n-1-j} v^{\mu}\cdot\Pi\cdot{}_{\mu}p_{x+j}$$

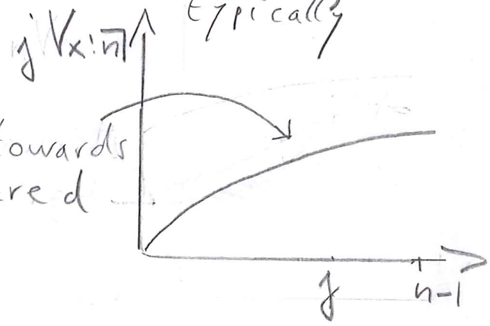
$$= \overset{=\Pi}{P_{x:\overline{n|}}}\cdot \ddot{a}_{x+j:\overline{n-j|}} \leftarrow \text{net sing. pr. of a temp. life annuity}$$

$$\Longrightarrow {}_jV_{x:\overline{n|}} := {}_jV = E[\underset{= {}_jZ - {}_jP}{{}_jL} \mid T>j] = E[{}_jZ \mid T>j] - E[{}_jP \mid T>j]$$

$$= A_{x+j:\overline{n-j|}} - P_{x:\overline{n|}}\cdot\ddot{a}_{x+j:\overline{n-j|}} \qquad (7.1.3)$$

$$\xrightarrow[\text{in (7.1.3)}]{j=n-1} \quad {}_{n-1}V_{x:\overline{n|}} = v - P_{x:\overline{n|}} \iff (1+i)\left({}_{n-1}V_{x:\overline{n|}} + P_{x:\overline{n|}}\right) = 1$$

$\longrightarrow$ interest earned on ${}_{n-1}V$ plus the premium is used to cover the sum insured (i.e. 1)



Ex. 7.1.3 (term insurance)

cash flow: benefits

$$0 \quad 1_{\{K=0\}} \quad 1_{\{K=1\}} \quad \ldots \quad 1_{\{K=n-1\}} \quad 0$$

$$0 \quad 1 \quad 2 \quad \ldots \quad t=j \quad \ldots \quad n \quad n+1 \quad \ldots$$

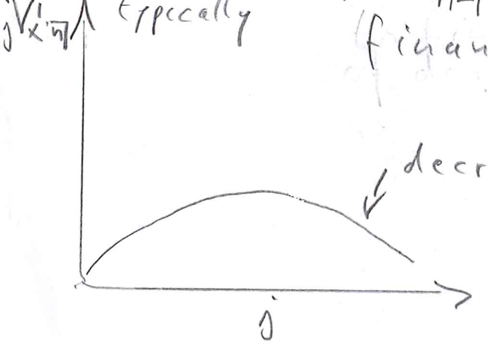
cash flow: premiums as in Ex. 7.1.2

$$\xrightarrow[\text{as in Ex. 7.1.2}]{\text{same reasoning}} \quad {}_jV^{1}_{x:\overline{n|}} := {}_jV = A^{1}_{x+j:\overline{n-j|}} - P^{1}_{x:\overline{n|}}\,\ddot{a}_{x+j:\overline{n-j|}} \qquad (7.1.4)$$

$A^{1}_{x+j:\overline{n-j|}} \Leftarrow$ NSP of a term insurance

$$\xrightarrow[\text{in (7.1.4)}]{j=n-1} \quad \left({}_{n-1}V^{1}_{x:\overline{n|}} + P^{1}_{x:\overline{n|}}\right) = A^{1}_{x+n-1:\overline{1|}} \quad (= v\cdot q_{x+n-1})$$

$\longrightarrow$ ${}_{n-1}V$ plus the premium is used to finance the NSP of a 1-year term insurance



(27)

7.2 Recursion formulas for ${}_jV$

Consider the gen. life insurance of Section 4.3 with benefits c_j and premiums Π_j

$\longrightarrow$ $$L = c_{K+1} v^{K+1} - \sum_{j=0}^{K} \Pi_j \cdot v^j$$

Rem. 7.2.1 : This model is quite general and includes various endowments and life annuities e.g. an endowment is obtained, if

and $c_1 = c_2 = \dots = c_n = 1$, $c_{n+1} = c_{n+2} = \dots = 0$

$\Pi_0 = \Pi_1 = \dots = \Pi_{n-1} = P_{x:\overline{n|}}$, $\Pi_n = -1$, $\Pi_{n+1} = \Pi_{n+2} = \dots = 0$

As in Ex. 7.1.2 one shows that (7.2.1)

$${}_kV = \sum_{j\geq 0} c_{k+j+1} v^{j+1} {}_jp_{x+k}\, q_{x+k+j} - \sum_{j\geq 0} \Pi_{k+j} v^j {}_jp_{x+k}$$

Using $\quad {}_jp_{x+k} = {}_hp_{x+k} \cdot {}_{j-h}p_{x+k+h}$, $j \geq h$

in (7.2.1), one finds that (7.2.2)

$${}_kV + \sum_{j=0}^{h-1} \Pi_{k+j} v^j {}_jp_{x+k} = \sum_{j=0}^{h-1} c_{j+k+1} v^{j+1} {}_jp_{x+k}\, q_{x+k+j} + {}_hp_{x+k} v^h {}_{k+h}V$$

$\xrightarrow[\text{in (7.2.2)}]{h=1}$ $${}_kV + \Pi_k = v\left(c_{k+1} q_{x+k} + {}_{k+1}V \cdot p_{x+k}\right)$$

$\Longleftrightarrow$ $${}_kV + \Pi_k = v\left({}_{k+1}V + (c_{k+1} - {}_{k+1}V)\, q_{x+k}\right) \qquad (7.2.3)$$

$\longrightarrow$ ${}_kV$ and Π_k are used to cover the PV (at $t=k$) of ${}_{k+1}V$ and the amount $c_{k+1} - {}_{k+1}V$ (needed at the end of the year), if the insured dies

$\longrightarrow$ $c_{k+1} - {}_{k+1}V$ <u>net amount at risk</u>

Further (7.2.3) admits the following <u>decomposition of Π_k</u> :

$$\Pi_k = \Pi_k^s + \Pi_k^r,$$

where $\quad \Pi_k^s = {}_{k+1}V \cdot v - {}_kV$ <u>savings premium</u> (used to increase the net. prem. reserves)

and $\quad \Pi_k^r = (c_{k+1} - {}_{k+1}V) \cdot v \cdot q_{x+k}$ <u>risk premium</u> (used to cover the net amount at risk)

Another reformulation of (7.2.3) gives

<u>Th. 7.2.2</u> (discrete-time version of Thiele's diff. eq.)

$$\Pi_k + d \cdot {}_{k+1}V = ({}_{k+1}V - {}_kV) + \Pi_k^r,$$

where $d := \frac{i}{1+i}$ is the <u>discount rate</u>, i.e. the int. rate credited at the beginning of each conversion period

$\longrightarrow$ prem. + int. on ${}_{k+1}V$ is used to cover the change of the net premium reserve and to finance the risk premium

7.3 Allocation of the total loss to policy years

Consider the whole life insurance of Section 7.2.
The total loss of the insurer during the year $k+1$ can be defined as

$$\Lambda_k = \begin{cases} 0, & \text{if } K \le k-1 \\ c_{k+1} \cdot v - ({}_kV + \Pi_k), & \text{if } K = k \\ {}_{k+1}V \cdot v - ({}_kV + \Pi_k), & \text{if } K \ge k+1 \end{cases} \qquad (7.3.1)$$

PV of c_{k+1} at $t=k$

PV of ${}_{k+1}V$ at $t=k$

$\Pi_k = \Pi_k^s + \Pi_k^r$ $\Longleftrightarrow$

$$\Lambda_k = \begin{cases} 0, & \text{if } K \le k-1 \\ -\Pi_k^r + (c_{k+1} - {}_{k+1}V) \cdot v, & \text{if } K = k \\ -\Pi_k^r, & \text{if } K \ge k+1 \end{cases} \qquad (7.3.2)$$

$\overset{\text{def. of } \Lambda_k}{\Longrightarrow}$

$$L = \sum_{k \ge 0} \Lambda_k v^k \qquad (7.3.3)$$

$\longrightarrow$ decomposition of the total loss into losses during the corresponding policy years

$\longrightarrow$ Th. 7.3.1 (Hattendorf's Theorem)

$$\operatorname{Cov}[\Lambda_k, \Lambda_j] = 0, \quad k \ne j$$

and

$$\operatorname{Var}[L] = \sum_{k \ge 0} v^{2k} \operatorname{Var}[\Lambda_k]$$

Proof: $E[\Lambda_k \mid K \ge k] \overset{(7.3.2)}{=} -\Pi_k^r q_{x+k} + \Pi_k^r - \Pi_k^r p_{x+k} = 0 \qquad (*)$

$\Rightarrow E[\Lambda_k] = E[\Lambda_k \mid K \ge k] \Pr(K \ge k) = 0$

$\Rightarrow \operatorname{Cov}[\Lambda_k, \Lambda_j] = E[\Lambda_k \cdot \Lambda_j] = E[\Lambda_k \cdot \Lambda_j \mid K \ge j] \Pr(K \ge j)$

$\underset{k<j}{\overset{(7.3.2)}{=}} -\Pi_k^r \underbrace{E[\Lambda_j \mid K \ge j]}_{\overset{(*)}{=} 0} \cdot \Pr(K \ge j) = 0$

$\Rightarrow \Lambda_k, \Lambda_j$ uncorrelated for $k < j$ $\Rightarrow$ proof.