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7.4 Technical gain

Assume the gen. life insurance of Section 7.2

$\longrightarrow \quad L = c_{K+1} v^{K+1} - \sum_{j=0}^{K} \Pi_j v^j$

Let i' be the investment yield (e.g. the interest of a pension fund) of a life company during year $k+1$

Then the technical gain at the end of the year is given by

$$G_{k+1} = \begin{cases} ({}_kV + \Pi_k)(1+i') - c_{k+1}, & \text{if } K = k \\ ({}_kV + \Pi_k)(1+i') - {}_{k+1}V, & \text{if } K \geq k+1 \end{cases} \quad (7.4.1)$$

accumulated int. gain on ${}_kV$ and Π_k

Rem. 7.4.1 (i) The technical gain is sometimes used for the purpose of "profit testing", i.e. the testing of the profit of a life product under the influence of various parameters of the model (i.e deathrate)

(ii) In addition technical gains based on the repurchase of an insurance, expense loadings (e.g. administration expenses) or taxes can be also considered.

$\longrightarrow$ decomposition of G_{k+1}:

1. method:

total loss during year $k+1$

$$G_{k+1} = ({}_kV + \Pi_k)(i' - i) - \Lambda_k (1+i) \quad (7.4.2)$$

investment gain

mortality gain, since Λ_k depends on K

2. method:

$$G_{k+1} = G^S_{k+1} + G^T_{k+1}, \quad (7.4.3)$$

where

gain from savings $\rightarrow G^S_{k+1} = ({}_kV + \Pi^S_k)(i' - i) = {}_{k+1}V \cdot v (i' - i)$

and

gain from the insurance $\rightarrow G^T_{k+1} = \begin{cases} \Pi^T_k (1+i') - (c_{k+1} - {}_{k+1}V), & K = k \\ \Pi^T_k (1+i'), & K \geq k+1 \end{cases}$

Rem. 7.4.2: G^S_{k+1} can be e.g. used to increase the benefits from time $t = k+1$ if $K \geq k+1$

$\longrightarrow \quad {}_{k+1}V + \underbrace{G^S_{k+1}}_{{}_{k+1}V \cdot v(i'-i)}$ should be equal to $E[{}_{k+1}\tilde{L} \mid T > k+1]$ (modified total loss)

$$\overset{(7.2.1)}{=} \sum_{j \geq 0} \tilde{c}_{k+1+j+1} v^{j+1} {}_jp_{x+k+1} q_{x+k+1+j} - \sum_{j \geq 0} \tilde{\Pi}_{k+1+j} v^j {}_jp_{x+k+1}$$

modif. benefits; modif. prem.

(7.2.1) $\rightarrow$ is possible, if e.g.

$\tilde{C}_{K+1+j+1} = v\cdot(1+i')\cdot C_{K+1+j+1}$ and $\tilde{\Pi}_{K+1+j} = v\cdot(1+i')\Pi_{K+1+j}$
for $j \geq 0$
If $K = \kappa$ the additional amount G^{S}_{K+1} could be provided

7.5 The continuous model

Consider the continuous version of the gen. life insurance of Sect. 7.2

$$\longrightarrow \quad L = C(T)\,v^{T} - \int_0^T \Pi(t)\,v^{t}\,dt$$

Now if we introduce m payments per year with conversion period $\frac{1}{m}$ in (7.2.1) and Th. 7.2.2 and let $m \to \infty$ we obtain:

$$V(t) := {}_tV = \int_0^\infty C(t+h)\,v^{h}\,{}_hp_{x+t}\cdot\mu_{x+t+h}\,dh - \int_0^\infty \Pi(t+h)\,v^{h}\,{}_hp_{x+t}\,dh \qquad (7.5.1)$$

The net prem. reserve $V(t)$ satisfies Thiele's diff. equation, i.e.

$$\Pi(t) + \delta V(t) = V'(t) + \Pi^{r}(t) \qquad (7.5.2)$$

Here $\Pi(t)$ can be decomposed as

$$\Pi(t) = \Pi^{S}(t) + \Pi^{r}(t),$$

where

$$\Pi^{S}(t) = V'(t) - \delta V(t)$$

is the savings premium
and

$$\Pi^{r}(t) = (C(t) - V(t))\,\mu_{x+t}$$

the risk premium

Ex. 7.6.1 : $C(t) \equiv 1$, $\Pi(t) \equiv 0$, $V(t) = \bar{A}_{x+t}$ (Sect. 4.4)
or $C(t) \equiv 0$, $\Pi(t) \equiv -1$, $V(t) = \bar{a}_{x+t}$ ← continuous life annuity

Non-Life insurance

31 8. Basic model for the total claim amount in a portfolio

The modeling of the total claim amount in a portfolio of insurance policies plays a central role in (non-) life insurance and serves e.g. as a tool for determining the amount of money paid by the insured in form of premiums to prevent the insurance company from going bankrupt.
important to hold balance between claims and premiums to avoid bankruptcy!

In the sequel we shall focus on a stoch. model for the total claim amount initiated by F. Lundberg (1903)

⟶ Model for the dynamics of homogenous insurance portfolios, that is for portfolios of insurance policies in the same "risk category" charaterized e.g. by same tariff groups (e.g. insurance for private cars, for commercial transport or fire insurance of wooden one-family houses)

⟶ Model set-up: The total claim amount at time t is modelled by the total (or aggregate) claim amount process $S(t)$, $t \geq 0$ given by

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i \geq 1} X_i 1_{[0,t]}(T_i), \quad t \geq 0, \quad (8.1)$$

where

T_i claim arrival times, that is the times when the claims occur

X_i claim size at time T_i, $i \in \mathbb{N}$

and

$N(t) \in \mathbb{N}_0$ claim number process given by

$$N(t) := \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0$$

(# = number of)

⟶ number of claims arriving by time t

(32) <u>model assumptions</u>:

$T_i, X_i, i \geq 1$ are modelled by r.v.'s on a prob. space $(\Omega, \mathcal{F}, P)$

such that

(i) $0 \leq T_1 \leq T_2 \leq \dots$

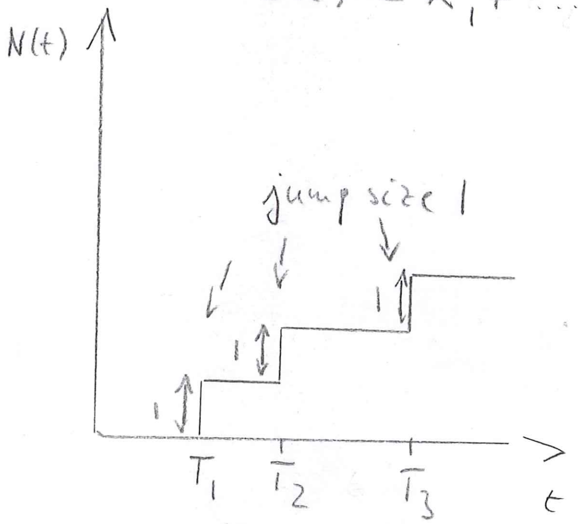
(ii) $X_i \geq 0, i \in \mathbb{N}$ and i.i.d $\leftarrow$ homogeneity of a portfolio

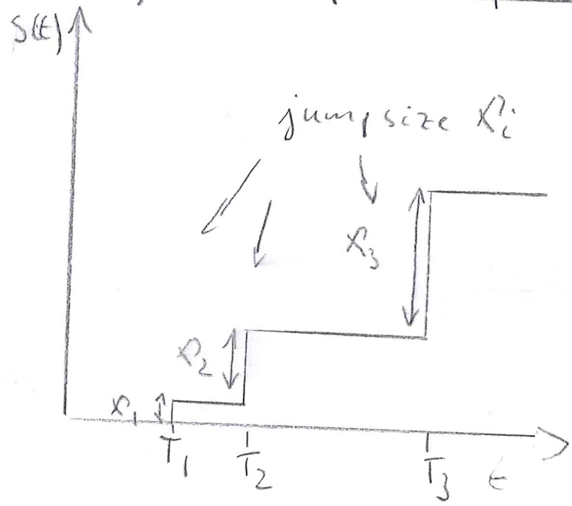
(iii) $(X_i)_{i \in \mathbb{N}}$ and $(T_i)_{i \in \mathbb{N}}$ mutually independent

Recall: $Y_1, \dots, Y_n$ r.v.'s indep. $\Longleftrightarrow$

$$P(Y_1 \leq t_1, \dots, Y_n \leq t_n) = P(Y_1 \leq t) \cdot \ldots \cdot P(Y_n \leq t)$$

<u>Rem</u>: $S(t) = X_1 + \dots + X_{N(t)}$ <u>compound process</u>





Model (8.1) gives rise to the <u>following problems</u>:

1. Find <u>realistic</u> and <u>mathematically tractable</u> stochastic models for the claim sizes X_i and the claim arrival times T_i

$\longrightarrow$ claim number proc. $N(t)$ described e.g. by a (mixed) Poisson process or a renewal proc. (see later)

or

$S(t)$ given by a compound Poisson process

2. <u>Properties of S and N</u>:

(i) distributions, moments and other distributional characteristics of S, N

(ii) Asymptotic behaviour of $S(t), N(t)$, that is e.g.: What is the average $\frac{N(t)}{t}$ or the distribution of $S(t)$ for large t?

3. Development of simulation methods for N and S, i.e. methods based on the power and memory of modern computers to approximate the distr. of N and S
   → e.g. Monte-Carlo approximation to S

4. How can be premiums, reserves or prices of insurance products determined on the basis of the properties of N and S?

→ Essential part of our agenda is to address the problems 1., 2., 3. and 4.

Let us first cope with 1., that is with various models for $N(t)$:

9. Models for the claim number process

9.1 $N(t)$ as Poisson process

Notation:

(i) $M \in \mathbb{N}_0$ random variable
We say M is Poisson distributed and write
$$M \sim \text{Pois}(\lambda)$$
for $\lambda > 0$, if
$$P(M = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \mathbb{N}_0$$

(ii) f: function on $[0,\infty)$
Define
$$f(s,t] = f(t) - f(s), \quad 0 \leq s < t < \infty$$

Def. 9.1.1 (Poisson process)
A stochastic proc. $N = (N(t))_{t \geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called Poisson process if

(1) $N(0) = 0$ (P-a.e.)

(2) N has independent increments:
$N(t_0, t_1] = N_{t_1} - N_{t_0}, \ldots, N(t_{n-1}, t_n] = N_{t_n} - N_{t_{n-1}}$
are indep. r.v.'s for all $0 = t_0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$

(3) $\exists$ non-decreasing right-continuous function $\mu : [0,\infty) \to [0,\infty)$ with $\mu(0)=0$ s.t.

$$\underset{N(t)-N(s)}{\overset{\|}{N(s,t]}} \sim \text{Pois}(\underset{\mu(t)-\mu(s)}{\overset{\|}{\mu(s,t]}})$$

for $0 \le s < t < \infty$

μ <u>mean value function of N</u>

(4) With probability 1 the <u>sample paths</u> of $(N(t,\omega))_{t\geq 0}$ are <u>càdlàg</u>, that is $\exists\ \tilde{\Omega} \in \mathcal{F}$ with $P(\tilde{\Omega})=1$ s.t. for all $\omega \in \tilde{\Omega}$

$$(t \mapsto N(t,\omega))$$

right-continuous with existing left limits

<u>Rem. 9.1.2</u>

(i) Interpr. of $\mu(s,t]$: expect. number of claim arrivals in the time period $(s,t]$

(ii) $(N(t_1), N(t_2), \dots, N(t_n))$

$= (N(t_1), N(t_1)+N(t_1,t_2], N(t_1)+N(t_1,t_2]+N(t_2,t_3], \dots, \sum_{i=1}^{n} N(t_{i-1},t_i])$

$= F(N(t_1,t_2], \dots, N(t_{n-1},t_n])$ with $t_1 < \dots < t_n$ and

$F(x_1,\dots,x_n) = (x_1, x_1+x_2, \dots, \sum_{i=1}^{n} x_i)$

$\longrightarrow$ function of indep. Poisson distr. r.v.'s

(iii) $N(t) = N(t)-N(0) = N(0,t] \sim \text{Pois}(\mu(0,t]) \overset{\mu(0)=0}{=} \text{Pois}(\mu(t))$

and $\mu(t) = E[N(t)] = \text{Var}[N(t)]$

<u>Rem. 9.1.3</u> (Simulation of a $X \sim \text{Pois}(\lambda)$)

Simulation is based on <u>random numbers of the uniform distribution on $[0,1]$</u>:

One way is e.g. the <u>congruential random number generator</u>, which generates rand. numbers according to the formula

$$R_n = \frac{Y_n}{m}, \qquad (*)$$

where

$$Y_{n+1} = (aY_n + b) \bmod (m+1), \quad Y_0 \in \{0,\dots,m-1\}$$

for "nice" $a,b,m \in \mathbb{N}_0$

; $20 \bmod 4 = 0$, $21 \bmod 4 = 1$, $22 \bmod 4 = 2$, ...

$p_j := P(X=j) = e^{-\lambda} \frac{\lambda^j}{j!}$

Def.

$F_0 = 0$, $F_{k+1} := \sum_{j=0}^{k} p_j$, $k \geq 0$

(35) $\longrightarrow$ $X \overset{d}{=} X^* := \sum_{j \geq 1} j \cdot 1_{\{F_{j-1} < U \leq F_j\}}$ (**)

U unif. distr. r.v. on $[0,1]$

Choose successively $U = R_1, U = R_2, U = R_3, \ldots$ in (*)

$\longrightarrow$ output of X^* gives a simulation of X?

$\longrightarrow$ simulation procedure for

$$S = \sum_{i=1}^{N} X_i$$

with $N \sim \text{Pois}(\lambda)$, X_i i.i.d., unif. distr., $N, (X_i)$ indep.

<u>9.1.1 Homogeneous Poisson process, Cramér-Lundberg model</u>

<u>Some notions:</u>

$\mu(t) = \lambda \cdot t$, $t \geq 0$, $\lambda > 0$

$\longrightarrow$ $N(t)$ <u>homogeneous Poisson proc.</u> (otherwise <u>inhomogenous</u>)

$\lambda = 1$

$\longrightarrow$ N <u>standard</u> hom. Poiss. proc.

$$\mu(s,t] = \int_s^t \lambda(y)\, dy, \quad s < t$$

for a function $\lambda \geq 0$

$\longrightarrow$ λ intensity function of N

$\longrightarrow$ non-constant intensity function implies that the intensity of the arrival of claims varies over time

$\longrightarrow$ N with such a λ captures <u>seasonal effects</u> observed e.g. in windstorm insurance

Choose in (8.1) N to be a homog. Poisson proc.

$\longrightarrow$ <u>Def. 9.1.1.1</u> (Cramér-Lundberg model)

(i) The claim arrivals $0 \leq T_1 \leq T_2 \leq \ldots$ are given by the jumptimes of a claim number process $N(t)$ modelled by a homog. Poisson proc.

(ii) The claim sizes X_i (at time T_i) are non-neg. and i.i.d.

(iii) (T_i) and (X_i) independent ($\Longrightarrow$ N and (X_i) indep.)

<u>Rem. 9.1.1.2</u> (i) $S(t)$ becomes a stoch. proc. with <u>independent and stationary increments</u>

"stationary" means: $S(t+h) - S(t) \overset{d}{=} S(h)$, $h, t \geq 0$

$\longrightarrow$ $S(t)$ <u>compound Poisson process</u>

$\longrightarrow$ <u>Lévy process</u> like e.g. the Brownian motion (See Exerc.)

(ii) $S(t)$ is a <u>Markov process</u>, that is

$$P(S(t_n) \in B \mid S(t_1), \dots, S(t_{n-1}))$$
$$= P(S(t_n) \in B \mid S(t_{n-1}))$$

for all $0 = t_0 < t_1 < \dots < t_n$ and (meas.) $B \subseteq [0, \infty)$

$\longrightarrow$ $S(t)$ process "without memory".

9.1.2 <u>Relations between the homogenous and inhom. Poisson process</u>

The inhom. Poisson proc. is a hom. Poiss. proc. under a <u>change of time</u>

$\longrightarrow$ <u>Prop. 9.1.2.1</u>:

Let μ be the mean value function of a Poisson proc. N and $\widehat{N}$ be a standard hom. Poiss. proc. (i.e. $\lambda = 1$). Then

(i) $(\widehat{N}(\mu(t)))_{t \geq 0}$ Poisson with mean value function μ

(ii) If μ continuous, increasing with $\lim\limits_{t \to \infty} \mu(t) = \infty$ then

$$(N(\mu^{-1}(t)))_{t \geq 0} \text{ stand. hom. Poisson proc.}$$

Here μ^{-1} denotes the inverse of μ

<u>Proof</u>: (i) It is sufficient to verify (1), (2), (3) in Def. 9.1.1 (Def. of Poiss. proc.)!

Def. $N(t) = \widehat{N}(\mu(t))$, $t \geq 0$

(1) $N(0) = \widehat{N}(\mu(0)) \overset{\mu(0)=0}{=} \widehat{N}(0) = 0$

(2) indep. increm.: Choose

$0 = t_0 < t_1 < \dots < t_n$

$\widehat{t}_i := \mu(t_i)$, $i = 0, \dots, n$ $\quad \xrightarrow[\text{increasing}]{\mu \text{ (strictly)}}$

$0 = \widehat{t}_0 < \widehat{t}_1 < \dots < \widehat{t}_n$

$\Longrightarrow$ $N(t_0, t_1] = \widehat{N}(\widehat{t}_0, \widehat{t}_1], \dots, N(t_{n-1}, t_n] = \widehat{N}(\widehat{t}_{n-1}, \widehat{t}_n]$ indep.

(3) $N(s,t] = N(t) - N(s) = \widehat{N}(\mu(t)) - \widehat{N}(\mu(s))$

$= \widehat{N}(\mu(s), \mu(t)] \sim \text{Pois}(\underbrace{\mu(t) - \mu(s)}_{\mu(s,t]})$

(ii) use the inverse μ^{-1} of μ

<u>Rem. 9.1.2.2</u> (Simulation of $(N(t_1),\dots,N(t_n))$)

Let $0=t_0<t_1<\dots<t_n$.

$(N(t_1),\dots,N(t_n)) = F(N(t_1,t_2],\dots,N(t_{n-1},t_n])$
$= F(\tilde{N}(\tilde{t}_1,\tilde{t}_2],\dots,\tilde{N}(\tilde{t}_{n-1},\tilde{t}_n])$,

where $\tilde{t}_i := \mu(t_i)$, $i=1,\dots,n$

and

$$F(x_1,\dots,x_n) = (x_1, x_1+x_2,\dots,\sum_{i=1}^n x_i)$$

$\longrightarrow$ $X_i := \tilde{N}(\tilde{t}_{i-1},\tilde{t}_i] \sim \text{Pois}(\tilde{t}_i-\tilde{t}_{i-1})$, $i=1,\dots,n$, indep.

Employ the rand. number generator of a unif. distr. (Rem. 9.1.3) n times to simulate an outcome of $(X_1,\dots,X_n)$.

Then use F to obtain $(N(t_1),\dots,N(t_n))$

<u>9.1.3 Homogeneous Poisson proc. as renewal process</u>

<u>Def. 9.1.3.1</u> (Renewal process)

Let $(N(t))_{t\geq 0}$ be a proc. s.t.

$$N(t) = \underset{\text{number of}}{\#}\{i\geq 1 : T_i \leq t\},\ t\geq 0,$$

where

$$T_0=0,\ T_n = W_1+\dots+W_n$$

for an i.i.d. sequence $(W_i)_{i\geq 1}$ of positive r.v.'s

Then $N(t)$ is called <u>renewal (counting) process</u>

The r.v.'s $W_i = T_i - T_{i-1}$ are referred to as <u>inter-arrival times</u>.

$\longrightarrow$ N hom. Poiss. proc.
N renewal proc. ? and distr. of W_i ?

$\longrightarrow$ <u>Th. 9.1.3.2</u> (N as renewal proc.)

(i) If N is a hom. Poiss. proc. with intensity λ and arrival times $0\leq T_1\leq T_2\leq\dots$ then N is a renewal proc. with i.i.d. $W_i \sim \text{Exp}(\lambda)$, $i\geq 1$, that is

$$P(W_i\leq x) = \int_0^x f(x)\,dx \qquad (9.1.3.1)$$

with density $f(x) = \lambda e^{-\lambda x}$, $x\geq 0$

38 (ii) Let N be a renewal proc. with i.i.d $W_i \sim Exp(\lambda)$. Then N is a hom. Poisson proc. with intensity $\lambda > 0$.

Proof: (i) It's sufficient to show that

$$\underbrace{P(T_1 \le x_1, \dots, T_n \le x_n)}_{=: \mathfrak{I}_1} = \underbrace{P(W_1 \le x_1, \dots, W_1 + \dots + W_n \le x_n)}_{=: \mathfrak{I}_2} \quad (*)$$

for all $x_i, i = 1, \dots, n, n \in \mathbb{N}$ for i.i.d $W_i \sim Exp(\lambda)$

$0 \le T_1 \le T_2 \le \dots$

$\longrightarrow$ w.l.o.g $0 \le x_1 \le x_2 \le \dots \le x_n$ in $(*)$

Recall: $F: \mathbb{R}^n \to \mathbb{R}$, $g(x_1, \dots, x_n)$ density of $(X_1, \dots, X_n)$, i.e.

$$P(X_1 \le x_1, \dots, X_n \le x_n) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} g(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

Then

$$E[F(X_1, \dots, X_n)]$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} F(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)\, dx_1 \dots dx_n \quad (**)$$

Consider $n = 2$

$$\mathfrak{I}_2 = E[1_{\{0 \le W_1 \le x_1,\, 0 \le W_1 + W_2 \le x_2\}}(W_1, W_2)]$$

$F(w_1, w_2) := 1_{\{0 \le w_1 \le x_1,\, 0 \le w_1 + w_2 \le x_2\}}(w_1, w_2)$, $g(w_1, w_2) = \lambda e^{-\lambda w_1} \lambda e^{-\lambda w_2}$

$$\overset{(**)}{\Longrightarrow} \mathfrak{I}_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(w_1, w_2) \cdot g(w_1, w_2)\, dw_1\, dw_2$$

On the other hand use

$$\{T_1 \le x_1, T_2 \le x_2\} = \{N(x_1) \ge 1, N(x_2) \ge 2\}, \text{Rem. } 9.1.2(ii), (**)$$

to verify that $\mathfrak{I}_1 = \mathfrak{I}_2$ (See Exerc. 6) $\Longrightarrow$ proof

(ii) See Mikosch

Rem. 9.1.3.2.:

(i) Since

$$S(t) = \sum_{i \ge 1} X_i 1_{[0,t]}(T_i), \quad t \ge 0 \quad (\text{see } 8.1)$$

$$T_i = W_1 + \dots + W_i$$

one can employ the indep. i.i.d. sequences (X_i) and (W_i) to simulate $S(t)$ (See later)

(ii) Since $W_i, i \ge 1$ possesses a density (See 9.3.1.1) we see that $W_i > 0, i \ge 1$ with prob. 1

$\Longrightarrow$ $0 < T_1 < T_2 < \dots$ with prob. 1

$\Longrightarrow$ $N(t)$ only has jumps of order 1 with prob. 1

39 What is the distr. of $(T_1,\dots,T_n)$ and $(W_1,\dots,W_n)$?

$\longrightarrow$ Prop. 9.1.3.3 : Let N be a (inhom.) Poisson proc. with $\mu(t)=\int_0^t \lambda(s)\,ds$, $\lambda(x)>0$ continuous. Then

(i) the density of $(T_1,\dots,T_n)$ is given by
$$f_{T_1,\dots,T_n}(x_1,\dots,x_n)=e^{-\mu(x_n)}\prod_{i=1}^n \lambda(x_i)\, 1_{\{0<x_1<\dots<x_n\}}$$
(ii) the density of $(W_1,\dots,W_n)$ takes the form
$$f_{W_1,\dots,W_n}(x_1,\dots,x_n)=e^{-\mu(x_1+\dots+x_n)}\prod_{i=1}^n \lambda(x_1+\dots+x_i),\ x_i\geq 0$$

Proof: (Consider e.g. the case $\lambda(x)\equiv\lambda>0$

(i) Use $(T_1,\dots,T_n)=(W_1,\dots,\sum_{i=1}^n W_i)$ and the property that $W_i\sim \mathrm{Exp}(\lambda)$ and i.i.d.

(ii) $f_{W_1,\dots,W_n}=\prod_{i=1}^n f_{W_i}$

$\Rightarrow$ proof

Now let us discuss an important property of the Poisson proc. which can be e.g. invoked to determine the distr. of the generalized total claim amount
$$S(t)=\sum_{i=1}^{N(t)} g(T_i,X_i),$$
where $g:\mathbb{R}^2\to\mathbb{R}$.

$\longrightarrow$ 9.1.4 The order statistics property of the Poisson process

Notation: Let $X_1,\dots,X_n$ be indep. r.v.'s

$\longrightarrow$ $X_{(j)}$ jth smallest of $X_1,\dots,X_n$

$\longrightarrow$ r.v.'s $X_{(1)}\leq\dots\leq X_{(n)}$ order statistics of $X_1,\dots,X_n$

Ex.: $X_{(1)}=\min(X_1,\dots,X_n)$, $X_{(n)}=\max(X_1,\dots,X_n)$

We are coming to a main result of our course:

Th. 9.1.4.1 (Order statistics property of N)

Let N be a Poisson proc. with $\mu(t)=\int_0^t\lambda(s)\,ds$, $\lambda(x)>0$ continuous and $T_1<T_2<\dots$ a.e.

Then
$$P(T_1\leq x_1,\dots,T_n\leq x_n\mid N(t)=n)=P(X_{(1)}\leq x_1,\dots,X_{(n)}\leq x_n)$$
for $X_1,\dots,X_n$ i.i.d r.v.'s with density $f(x)=\dfrac{\lambda(x)}{\mu(t)}$, $0<x\leq t$.

Moreover
$$P(T_1\leq x_1,\dots,T_n\leq x_n\mid N(t)=n)=\int_0^{x_1}\dots\int_0^{x_n} f_{T_1,\dots,T_n}(x_1,\dots,x_n\mid N(t)=n)\,dx_1\dots dx_n,$$
cond. density

where
$$f_{T_1,\dots,T_n}(x_1,\dots,x_n\mid N(t)=n)$$
$$=f_{X_{(1)},\dots,X_{(n)}}(x_1,\dots,x_n)=\begin{cases}\dfrac{n!}{(\mu(t))^n}\prod_{i=1}^n\lambda(x_i), & 0<x_1<\dots<x_n<t\\ 0, & \text{else}\end{cases}$$

(40) Recall: $P(A|B) \overset{def.}{=} \frac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$, A, B events

Proof: Let us show that

$$\lim_{h_i \downarrow 0, i=1,\dots,n} \frac{P(T_1 \in (x_1, x_1+h_1], \dots, T_n \in (x_n, x_n+h_n] \mid N(t)=n)}{h_1 \cdots h_n}$$

$$= \frac{n!}{(\mu(t))^n} \prod_{i=1}^{n} \lambda(x_i), \quad 0 < x_1 < \dots < x_n < t \qquad (*)$$

Choose $h_i, i=1,\dots,n$ small enough s.t. $(x_i, x_i+h_i], i=1,\dots,n$ are disjoint

Consider the case $n=2$.

Then

$$\{T_1 \in (x_1, x_1+h_1], T_2 \in (x_2, x_2+h_2], N(t)=2\}$$
$$= \{N(0,x_1]=0, N(x_1,x_1+h_1]=1, N(x_1+h_1, x_2]=0, N(x_2,x_2+h_2]=1, N(x_2+h_2,t]=0\} \qquad (**)$$

Recall: $N(a,b]$ = number of claims between a and b

$$\overset{(**)}{\Longrightarrow} P(T_1 \in (x_1,x_1+h_1], T_2 \in (x_2,x_2+h_2], N(t)=2)$$
$$= P(N(0,x_1]=0, N(x_1,x_1+h_1]=1, N(x_1+h_1,x_2]=0, N(x_2,x_2+h_2]=1, N(x_2+h_2,t]=0)$$
$$\overset{\text{indep. increm. of } N}{=} P(N(0,x_1]=0)\, P(N(x_1,x_1+h_1]=1) \cdot P(N(x_1+h_1,x_2]=0) \cdot P(N(x_2,x_2+h_2]=1) \cdot P(N(x_2+h_2,t]=0)$$
$$\overset{\text{increm. Poisson distr.}}{=} e^{-\mu(x_1)} \cdot e^{-\mu(x_1,x_1+h_1]} \mu(x_1,x_1+h_1]\, e^{-\mu(x_1+h_1,x_2]} \cdot e^{-\mu(x_2,x_2+h_2]} \mu(x_2,x_2+h_2] \cdot e^{-\mu(x_2+h_2,t]}$$
$$= e^{-\mu(t)} \mu(x_1,x_1+h_1) \cdot \mu(x_2,x_2+h_2]$$

$$\Rightarrow P(T_1 \in (x_1,x_1+h_1], T_2 \in (x_2,x_2+h_2] \mid N(t)=2)/h_1 \cdot h_2$$
$$= P(T_1 \in (x_1,x_1+h_1], T_2 \in (x_2,x_2+h_2], N(t)=2) / h_1 \cdot h_2 \cdot P(N(t)=2)$$
$$= \frac{2!}{(\mu(t))^2} \cdot \underbrace{\frac{\mu(x_1,x_1+h_1]}{h_1}}_{\to \lambda(x_1)} \cdot \underbrace{\frac{\mu(x_2,x_2+h_2]}{h_2}}_{\to \lambda(x_2)}$$
$$\xrightarrow[h_1,h_2 \downarrow 0]{} \frac{2!}{(\mu(t))^2} \lambda(x_1)\lambda(x_2) = (*) \text{ for } n=2$$

Similarly $(*)$ can be established for $(x_i, x_i - h_i], i=1,\dots,n$