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$\Longrightarrow P(T_1 \le x_1, \dots, T_n \le x_n \mid N(t)=n)$ has a density given by $(*)$

The fact that

$$f_{X_{(1)},\dots,X_{(n)}} = (*)$$

follows from the next Lemma $\Longrightarrow$ proof

<u>Lemma 9.1.4.2</u> (Joint density of order statistics)

Let $X_i, i \ge 1$ be i.i.d with density f. Then

$$f_{X_{(1)},\dots,X_{(n)}}(x_1,\dots,x_n) = n! \prod_{i=1}^{n} f(x_i) \cdot 1_{\{x_1 < \dots < x_n\}}$$

<u>Proof</u>: See e.g. Mikosch, p. 28

<u>Ex. 9.1.4.3</u> (Order statistics property of hom. N)

hom. $N \longrightarrow \mu(t) = \lambda t, \lambda > 0$

$\overset{\text{Th. 9.1.4.1}}{\Longrightarrow}$ $f_{T_1,\dots,T_n}(x_1,\dots,x_n \mid N(t)=n) = n!\, t^{-n}, \quad 0 < x_1 < \dots < x_n < t$

Let $U_1,\dots,U_n$ be i.i.d with $U_i \sim U(0,t)$, that is

$$P(U_i \le x) = \frac{x}{t}, \quad x \in [0,t]$$

$\overset{\text{Th. 9.1.4.1}}{\Longrightarrow}$

$$f_{U_{(1)},\dots,U_{(n)}}(x_1,\dots,x_n) = f_{T_1,\dots,T_n}(x_1,\dots,x_n \mid N(t)=n)$$

indep. of λ!

<u>Ex. 9.1.4.4</u> (applied to symmetric functions)

function g on $\mathbb{R}^n$ <u>symmetric</u> $\Longleftrightarrow$ $g(x_1,\dots,x_n) = g(x_{\pi(1)},\dots,x_{\pi(n)})$ for all permutations π of $\{1,\dots,n\}$

Ex.: $g = \prod_{i=1}^{n} x_i$ or $g = \sum_{i=1}^{n} x_i$

$\overset{\text{Th. 9.1.4.1}}{\Longrightarrow}$

$$\underbrace{(g(T_1,\dots,T_n) \mid N(t)=n)}_{\text{distr. of } g(T_1,\dots,T_n) \text{ given } N(t)=n} \overset{d}{=} g(X_{(1)},\dots,X_{(n)}) = g(X_1,\dots,X_n)$$

Using Th. 9.1.4.1 and the last two examples give (see Mikosch)

<u>Prop. 9.1.4.5</u> (Distr. of a generalized total claim amount)

Let N be a hom. Poisson proc. with $\lambda > 0$.

Then for all $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$:

$$\tilde{S}(t) := \sum_{i=1}^{N(t)} g(T_i, X_i) \overset{d}{=} \sum_{i=1}^{N(t)} g(t \cdot U_i, X_i),$$

where

$$(U_i) \text{ i.i.d } U(0,1) \text{ indep. of } (X_i) \text{ and } (T_i)$$

indep.

Ex. 9.1.4.6 : Present value of the claim payments in $S(t)$ is given by
$$\tilde{S}(t) := \sum_{i=1}^{N(t)} e^{-rT_i} X_i$$

$\longrightarrow \quad g(T, X) = e^{-rT} X$

Ex. 9.1.4.7 : Modelling arrivals of incurred but not reported claims :

reality : claim i is rather reported at $T_i + V_i$, $V_i \geq 0$ than at T_i with delay V_i

$\longrightarrow$ # of claims reported up to time t is given by
$$\tilde{N}(t) := \sum_{i=1}^{N(t)} 1_{[0,t]}(T_i + V_i)$$

Choose X_i as V_i in Prop. 9.1.4.5

9.2 N as a renewal process

Recall Def. 9.1.3.1:

$W_i \geq 0, i \geq 1$ i.i.d, $T_0 = 0$, $T_n = W_1 + \dots + W_n$, $n \geq 1$

$\longrightarrow$ $N(t) = \#\{i \geq 1 : T_i \leq t\}$, $t \geq 0$

renewal process

$\longrightarrow$ generalization of the hom. Poisson proc. with i.i.d $W_i \sim Exp(\lambda)$

Rem.: The name "renewal process" comes from applications to model the number of renewals $N(t)$ of a technical device (e.g. light bulbs in a factory) up to time t.

Why a renewal process instead of a hom. Poisson proc. ?

$\longrightarrow$ It's characteristic e.g. for windstorm insurance that inter-arrival may become very large

$\longrightarrow$ modelling of such W_i not captured by $Exp(\lambda)$

$\longrightarrow$ distributions with heavier tails like e.g. the log-normal or Pareto distr. are more appropriate

Recall: Pareto distr. $\Longleftrightarrow$ density f given by

$$f(x) = \begin{cases} 0, & x \leq x_0 \\ \frac{a}{x_0}\left(\frac{x_0}{x}\right)^{a+1}, & x > x_0 \end{cases}$$

for x_0 and $a > 0$ constants

We shall discuss asymptotic properties of the renewal proc. $N(t)$, which play a central role in insurance business!

Theorem 9.2.1 (SLLN for the renewal proc.)
Let $W_i, i \geq 1$ be the inter-arrival times of a renewal proc. $N(t)$ with $E[W_1] = \frac{1}{\lambda}$, $\lambda > 0$.

(44) Then the SLLN of $N(t)$ holds in the following sense:

$$\frac{N(t)}{t} \xrightarrow[t\to\infty]{} \lambda$$

Proof: Idea: SLLN applied to (W_i)

By def. of N we have that

$$N(t)=n \iff T_n \le t < T_{n+1}, \quad n\in \mathbb{N}_0$$

$\Longrightarrow$ $T_{N(t)} \le t < T_{N(t)+1}$

$\Longrightarrow$ "sandwich inequality":

$$\frac{T_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{T_{N(t)+1}}{N(t)+1}\,\frac{N(t)+1}{N(t)} \qquad (*)$$

On the other hand the SLLN gives

$$\frac{1}{n}T_n = \frac{1}{n}\sum_{i=1}^{n} W_i \longrightarrow E[W_1]=\frac{1}{\lambda}$$

(i.i.d)

$\Longrightarrow$ $T_n \to \infty$ for $n\to\infty$

$\Longrightarrow$ $N(t)\to\infty$ for $t\to\infty$

$\Longrightarrow$ $\frac{T_{N(t)}}{N(t)}, \frac{T_{N(t)+1}}{N(t)+1} \xrightarrow[t\to\infty]{} E[W_1]=\frac{1}{\lambda}$

$\overset{(*)}{\Longrightarrow}$ $\frac{t}{N(t)} \longrightarrow \frac{1}{\lambda}$ $\Longrightarrow$ proof.

We shall study the mean behaviour of $E[N(t)]$.

$\longrightarrow$ Theorem 9.2.2 (Elementary renewal theorem)

If $E[W_1]=\frac{1}{\lambda}$ for $\lambda>0$ then

$$\frac{E[N(t)]}{t} \xrightarrow[t\to\infty]{} \lambda$$

Proof: Idea: Wald's identity

$$W_i^{(b)} := \min(W_i, b), \quad T_i^{(b)} := W_1^{(b)}+\dots+W_i^{(b)}, \quad i\ge 1,\ b>0$$

$\longrightarrow$ truncated W_i, T_i

$$N_b(t) := \#\{i\ge 1 : T_i^{(b)} \le t\}, \quad t\ge 0$$

$\overset{\text{Def. 9.1.3.1}}{\Longrightarrow}$ $N_b(t)$ renewal proc.

$$T_n \ge T_n^{(b)} \Longrightarrow N_b(t) \ge N(t)$$

$$\Longrightarrow \overline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \le \overline{\lim_{t\to\infty}} \frac{E[N_b(t)]}{t} \qquad (+)$$

Now use Wald's identity to obtain that

$$E[T^{(b)}_{N_b(t)+1}] = E[N_b(t)+1]\cdot E[W_1^{(b)}] \qquad (++)$$

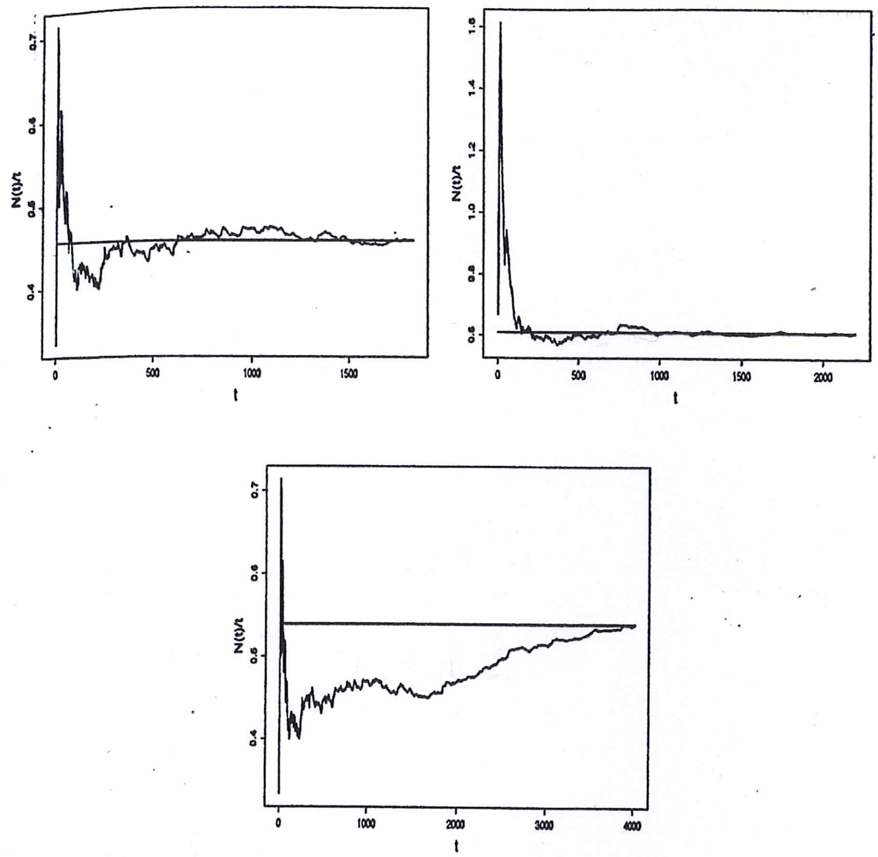

Figure 2.2.9 *Visualization of the validity of the strong law of large numbers for the arrivals of the Danish fire insurance data* 1980 – 1990; *see Section 2.1.7 for a description of the data.* Top left: *The ratio* $N(t)/t$ *for* 1980 – 1984, *where* $N(t)$ *is the claim number at day* t *in this period. The values cluster around the value* 0.46 *which is indicated by the constant line.* Top right: *The ratio* $N(t)/t$ *for* 1985 – 1990, *where* $N(t)$ *is the claim number at day* t *in this period. The values cluster around the value* 0.61 *which is indicated by the constant line.* Bottom: *The ratio* $N(t)/t$ *for the whole period* 1980 – 1990, *where* $N(t)$ *is the claim number at day* t *in this period. The graph gives evidence about the fact that the strong law of large numbers does not apply to* N *for the whole period. This is caused by an increase of the annual intensity in* 1985 – 1990 *which can be observed in Figure 2.1.21. This fact makes the assumption of iid inter-arrival times over the whole period of* 11 *years questionable. We do, however, see in the top graphs that the strong law of large numbers works satisfactorily in the two distinct periods.*

(45) By def. we have that

$$T^{(b)}_{N_b(t)} = W_1^{(b)} + \dots + W^{(b)}_{N_b(t)} \le t$$

$$\Rightarrow T^{(b)}_{N_b(t)+1} = T^{(b)}_{N_b(t)} + W^{(b)}_{N_b(t)+1} \le t+b \qquad (+++)$$

$$\Rightarrow \overline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \overset{(+),(++)}{\le} \overline{\lim_{t\to\infty}} \frac{E[T^{(b)}_{N_b(t)+1}]}{t\,E[W_1^{(b)}]} \overset{(+++)}{\le}$$

$$\overline{\lim_{t\to\infty}} \frac{t+b}{t\,E[W_1^{(b)}]} = \frac{1}{E[W_1^{(b)}]} \xrightarrow[b\to\infty]{} \frac{1}{E[W_1]} = \lambda$$

$W_1^{(b)} \searrow W_1$ for $b\to\infty$

One also shows that

$$\underline{\lim_{t\to\infty}} \frac{E[N(t)]}{t} \ge \lambda$$

$$\Rightarrow \lim_{t\to\infty} \frac{E[N(t)]}{t} = \lambda \qquad \Rightarrow \text{proof}$$

Rem 9.2.3: (i) Refinement of Th. 9.2.2: Blackwell's renewal theorem:

$$E[N(t,t+h]] \xrightarrow[t\to\infty]{} \lambda h \quad \text{for all } h>0$$

(ii) Consider Exerc. 6, Prob. 1, where the annual intensities $\hat{\lambda}$ of fire losses between 1980 and 1990 vary. → Observation:

$$\frac{N(t)}{t} \xrightarrow[t\to\infty]{} \lambda \quad \text{fails in this period (see slide)}$$

→ Deficiency of the renewal model: i.i.d assumption on (W_i) is not always appropriate (especially e.g. in insurances where seasonal effects have to be taken into account)

Prop. 9.2.4 (Asymptotic behaviour of $Var[N(t)]$)
If $Var[W_1] < \infty$, then

$$\lim_{t\to\infty} \frac{Var[N(t)]}{t} = \frac{Var[W_1]}{(E[W_1])^3}$$

Theorem 9.2.5 (CLT for $N(t)$)
If $Var[W_1] < \infty$, then

$$(Var[W_1]\cdot(E[W_1])^{-3}\cdot t)^{-\frac{1}{2}}(N(t)-\lambda t) \xrightarrow[t\to\infty]{d} Y \sim \mathcal{N}(0,1)$$

For the proofs see e.g. Embrechts et al. (1997): Modelling Extremal Events for Insurance and Finance. Springer.

Rem 9.2.6: Prop. 9.2.4 shows that $Var[W_1]\cdot E[W_1]^{-3} t$ can be replaced by $Var[N(t)]$

$$\Rightarrow N(t) \sim \mathcal{N}(\lambda\cdot t, Var[N(t)]) \quad \text{for large } t$$

9.3 N as mixed Poisson process

→ generalization of the inhom. Poisson proc. with deterministic mean value function $\mu(t)$ to a inhom. Poisson proc. with "randomized" $\mu(t)$

→ Def. 9.3.1 (Mixed Poisson process)
Let $\tilde{N}$ be a standard hom. Poisson proc. and μ a mean value function of a Poisson proc. Further let $\theta > 0$ be a (non-constant) r.v. indep. of $\tilde{N}$. Then

$$N(t) := \tilde{N}(\hat{\mu}(t)), \quad t \geq 0 \qquad (9.3.1)$$

with "randomized" $\mu(t)$, that is

$$\hat{\mu}(t) := \theta \cdot \mu(t)$$

is called mixed Poisson process with mixing variable θ.

→ Ex. 9.3.2:
$\mu(t) = t$, θ Gamma distributed with parameter α, λ, that is θ has density

$$f_\theta(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0, \alpha, \lambda > 0$$

→ $N(t)$ is negative binomially distr. with param p, v, that is

$$P(N(t) = k) = \binom{v+k+1}{k} p^v (1-p)^k, \quad k \in \mathbb{N}_0, \ p \in (0,1), v > 0$$

Here: $p = \lambda/(t+\lambda)$, $v = \alpha$

Why N instead of a inhom. Poisson proc. ?

→ For example in windstorm insurance seasonality effect is a single influencing factor which is captured by a deterministic $\mu(t)$

However e.g. a portfolio of car insurance policies is exposed to different influencing factors coming from the "individual history" of the policyholders (e.g. driving skills, age, ...)

→ Her each outcome $\hat{\mu}(t) = \theta(\omega)\mu(t)$, $\omega \in \Omega$ stands for the "individual history" of the policyholders

→ In general: The "randomized" $\mu(t)$, that is $\hat{\mu}(t)$ represents the entirety of different influencing factors

→ Generalization of the mixed Poisson proc.: Cox process

→ $\hat{\mu}(t)$ in (9.3.1) stoch. proc. which is non-decreasing in t and indep. of $\tilde{N}$

Properties of the mixed Poisson proc.:

Properties diff. from the inhom. Poisson proc.:

(i) Over-dispersion:

$$\mathrm{Var}[N(t)] \gneqq E[N(t)],$$

if $\mathrm{Var}[\theta] < \infty$, $\mu(t) > 0$ for $t > 0$ (See Exercises)

Note: $\mathrm{Var}[\tilde{N}(t)] = E[\tilde{N}(t)]$! for inhom. Poisson proc. $\tilde{N}(t)$

(ii) $N(t)$ has in general no independent increments

(iii) $N(t)$ is in general not Poisson distr.

Properties shared with the inhom. Poisson proc.

(i) $N(t)$ Markov proc., that is a proc. "without memory". See Rem. 9.1.1.2

(ii) Order statistics property:

order sample

$$(T_1, \dots, T_n \mid N(t) = n) \overset{d}{=} (X_{(1)}, \dots, X_{(n)}) \quad (9.3.2)$$

for arrival times $0 < T_1 < T_2 < \dots$ and the order sample of i.i.d $X_1, \dots, X_n$ with density $\frac{\lambda(x)}{\mu(t)}$, $0 \le x \le t$,

if $\mu(t) = \int_0^t \lambda(u)\,du$ with continuous $\lambda(u) > 0$

See Th. 9.1.4.1

10. Total claim amount $S(t)$

Basic model:

$$S(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0,$$

where N is indep. of the i.i.d X_i claim sizes

→ S Cramér-Lundberg model, if N hom. Poisson proc.

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S <u>renewal model</u>, if N renewal process

<u>Our programme</u>:

1. Study of the asymptotic growth properties of $S(t)$ as a basis for premium calculation
2. Premium calculation principles
3. Claim size distributions
   $\longrightarrow$ modelling of large and small claims
4. Distributional properties of $S(t)$
5. Numerical methods to evaluate the distr. of $S(t)$
   $\longrightarrow$ formula of Panjer
6. Approximation of the distr. of $S(t)$
   $\longrightarrow$ CLT or Monte Carlo techniques
7. Reinsurance

10.1 Order of magnitude of $S(t)$

In order to avoid ruin in a portfolio the insurance needs to determine <u>premiums</u> based on the order of magnitude of $S(t)$

$\longrightarrow$ Methods for measuring the size of $S(t)$ are based e.g. on $E[S(t)]$, $Var[S(t)]$, SLLN and CLT for $S(t)$

$\longrightarrow$ <u>Prop. 10.1.1</u> (Growth properties of $E[S(t)]$ and $Var[S(t)]$)

Consider the renewal model for $S(t)$.

(i) If $E[W_1] = \frac{1}{\lambda}$ for a $\lambda > 0$ and $E[X_1] < \infty$ then

$$\frac{E[S(t)]}{t} \xrightarrow[t\to\infty]{} \lambda E[X_1]$$

(ii) If $Var[W_1]$, $Var[X_1] < \infty$ then

$$\frac{Var[S(t)]}{t} \xrightarrow[t\to\infty]{} \lambda [Var[X_1] + Var[W_1]\lambda^2 (E[X_1])^2]$$

<u>Rem. 10.1.2</u>:

(i) Cramér-Lundberg model $\longrightarrow$
$E[S(t)] = \lambda \cdot t\, E[X_1]$ and $Var[S(t)] = \lambda \cdot t\, E[X_1^2]$

(ii) Using e.g. the equivalence principle to determine the premium payments $P(t)$ one observes from (i) in Prop. 10.1.1 that
$P(t)$ is (roughly) an increasing linear function

As a basis of premium calculation we shall study asymptotic properties of $S(t)$ with respect to the SLLN and CLT for $S(t)$

→ Theorem 10.1.3 (SLLN and CLT for $S(t)$ in the renewal model)
Assume that $S(t)$ is described by the renewal model.

(i) If $E[W_1] = \frac{1}{\lambda}$, $\lambda > 0$ and $E[X_1] < \infty$ then the SLLN for $S(t)$ holds, i.e

$$\frac{S(t)}{t} \xrightarrow[t \to \infty]{} \lambda E[X_1]$$

with prob. 1

(ii) If $Var[W_1]$, $Var[X_1] < \infty$ then $S(t)$ satisfies the CLT, i.e

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{S(t) - E[S(t)]}{\sqrt{Var[S(t)]}} \leq x \right) - \Phi(x) \right| \xrightarrow[t \to \infty]{} 0$$

Proof: (i) Rewrite

$$\frac{S(t)}{t} = \frac{S(t)}{N(t)} \frac{N(t)}{t} \qquad (+)$$

a) $\frac{N(t)}{t}$: The SLLN for the renewal proc. (Th. 9.2.1) gives

$$\frac{N(t)}{t} \xrightarrow[t \to \infty]{} \lambda$$

$$\Rightarrow \quad N(t) \xrightarrow[t \to \infty]{} \infty \quad \text{with prob. 1} \qquad (*)$$

b) $\frac{S(t)}{N(t)}$: $S_n = \sum_{i=1}^{n} X_i$

$$\overset{\text{SLLN}}{\Longrightarrow} \quad \frac{1}{n} S_n \xrightarrow[n \to \infty]{} E[X_1]$$

because of (*) and $N(t) \in \mathbb{N}_0$ we see that

$$\frac{S(t)}{N(t)} = \frac{S_j}{j}\Big|_{j = N(t)} \xrightarrow[t \to \infty]{} E[X_1]$$

$$\overset{(+)}{\Longrightarrow} \quad \frac{S(t)}{t} \rightarrow E[X_1] \cdot \lambda \quad \Rightarrow \text{proof}$$

(ii) See e.g. Embrechts et. al. : Modelling Extremal Events for Insurance and Finance. Springer (1997).

Rem. 10.1.4 : Th. 10.1.3 (ii) shows that
$S(t) \sim \mathcal{N}(E[S(t)], Var[S(t)])$ approximately for large t

10.2 Classical premium calculation principles

Basic problem in insurance:

How should be premiums determined to cover losses over time?

Some attempts to solve this problem are given by the following premium calculation principles:

1. Expected value principle: Denote by $p(t)$ the premium income.

From Prop. 10.1.1 and Th. 10.1.3 (SLLN for $S(t)$) we see that

$$S(t) \approx E[S(t)] \approx \lambda E[X_1] \cdot t \text{ for large } t \quad (*)$$

$\longrightarrow$ 1. case: $p(t) < E[S(t)] \Longrightarrow p(t) - S(t) < 0$ for large t

2. case: $p(t) > E[S(t)] \Longrightarrow p(t) - S(t) > 0$ for large t

2. case $\Longrightarrow$ It's reasonable to choose $p(t)$ by

$$p(t) = p_{EV}(t) := (1+\delta) E[S(t)]$$

for a safety loading $\delta > 0$, which reflects the assumed risk

Deficiency of $p_{EV}(t)$:

Too large δ makes the insurance less competitive

$\Longrightarrow$ number of insurance contracts decreases

$\Longrightarrow$ SLLN for $S(t)$ fails

However for a "feasible" risk analysis of insurance portfolios the insurance company relies on the SLLN

$\longrightarrow$ δ has to be chosen "reasonably"

2. Equivalence principle:

A theoretical reasonable premium price is e.g. provided by the following requirements

(i) Non-negative loading: $p(t) \geq E[S(t)]$

(ii) Consistency: premium for $S(t)+c$ is $p(t)+c$

(51)

(iii) <u>Additivity</u>: total claims $S(t)$, $S'(t)$ indep. with premiums $p(t)$, $p'(t)$ resp.

$\longrightarrow$ $p(t)+p'(t)$ premium for $S(t)+S'(t)$

(iv) <u>Proportionality</u>: $c \cdot p(t)$ premium for $c \cdot S(t)$, $c>0$

If
$$p(t) = E[S(t)] \quad \text{(Equivalence principle)}$$
then (i), …, (iv) are fulfilled

$\longrightarrow$
$$p(t) = p_{Net}(t) := E[S(t)]$$
"fair market premium"

$\longrightarrow$ total loss of the insurance company becomes zero in average

<u>Deficiency of $p_{Net}(t)$</u>:

a) The CLT for $S(t)$ (Th. 10.1.3) shows that
$$P\left(|S(t) - E[S(t)]| \geq \sqrt{Var[S(t)]} \right) > 0$$
for large t

$\Longrightarrow$ $S(t)$ deviates from $E[S(t)]$ at the order of $\sqrt{Var[S(t)]}$ for large t with positive probability

b) $p_{Net}(t)$ <u>leads to ruin</u>, i.e.
$$P\left(u + p_{Net}(t) - S(t) < 0 \text{ for some } t>0 \right) = 1$$
for all initial capital $u>0$ (see later)

$\Longrightarrow$ $p_{Net}(t)$ very unwise premium strategy

3. <u>Variance principle</u>:
$$p_{Var}(t) := E[S(t)] + \alpha \, Var[S(t)], \quad \alpha > 0$$

Choose e.g. $\alpha = \delta \dfrac{E[X_1]}{Var[X_1] + Var[W_1]\lambda^2 (E[X_1])^2}$

Prop. 10.1.1 $\Longrightarrow$ $\dfrac{p_{Var}(t)}{p_{EV}(t)} \xrightarrow[t\to\infty]{} 1 \quad \Longrightarrow \quad p_{Var}(t)$ equivalent to $p_{EV}(t)$

4. <u>Standard deviation principle</u>:
$$p_{SD}(t) := E[S(t)] + \alpha \sqrt{Var[S(t)]}, \quad \alpha > 0$$

<u>Motivation</u>:
$$P(S(t) - p_{SD}(t) \leq x) \xrightarrow[t\to\infty]{} \Phi(\alpha), \quad x \in \mathbb{R}$$
comes from the fact that
$$\frac{S(t)-E[S(t)]}{\sqrt{Var[S(t)]}} - \frac{x}{\sqrt{Var[S(t)]}} \xrightarrow[t\to\infty]{d} N(0,1) \quad \text{(Th 10.1.3)}$$

<u>Advantage of $p_{SD}(t)$</u>: $\dfrac{p_{SD}(t)}{p_{Net}(t)} = 1 + \alpha \underbrace{\dfrac{\sqrt{Var[S(t)]}}{E[S(t)]}}_{\xrightarrow[t\to\infty]{} 0} \xrightarrow[t\to\infty]{} 1$

$\Longrightarrow$ $p_{SD}(t) < p_{EV}(t)$, $p_{Var}(t)$ for large t $\Longrightarrow$ premiums can be offered at a lower price

(52)

10.3 Claim size distributions

What is a realistic model for claim size distributions?

The goodness of fit of claim size data to the model can be assessed by using e.g. QQ-plots or mean excess plots

→ 10.3.1 QQ-plots

Def. 10.3.1.1 (Quantile function)
The generalized inverse of the distr. function F, i.e.

$$F^{\leftarrow}(t) := \inf\{x \in \mathbb{R} : F(x) \geq t\}, \quad 0 < t < 1$$

is called the quantile function of F

Rem. 10.3.1.2 (i) $x_t := F^{\leftarrow}(t)$ t-quantile of F
(ii) F strictly increasing $\Rightarrow F^{\leftarrow}(t) = F^{-1}(t)$ (e.g. $F = \Phi$)

Def. 10.3.1.3 (Empirical distribution function)
Let $X_1, \ldots, X_n$ be i.i.d r.v.'s. Then

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty, x]}(X_i), \quad x \in \mathbb{R}$$

is called empirical distribution function of the sample $X_1, \ldots, X_n$

→ $F_n(x)$ distribution function, since

(i) $\lim_{x \to -\infty} F_n(x) = 0$ and $\lim_{x \to \infty} F_n(x) = 1$

(ii) $F_n(x) \leq F_n(y)$ for $x \leq y$

(iii) $\lim_{y \downarrow x} F_n(y) = F_n(x)$ for all $x \in \mathbb{R}$

As in Sect. 9.1.4 denote by

$$X_{(1)} \leq \ldots \leq X_{(n)}$$

the order statistics of $X_1, \ldots, X_n$
i.e. $X_{(j)}$ is the jth smallest of $X_1, \ldots, X_n$

In the sequel assume that

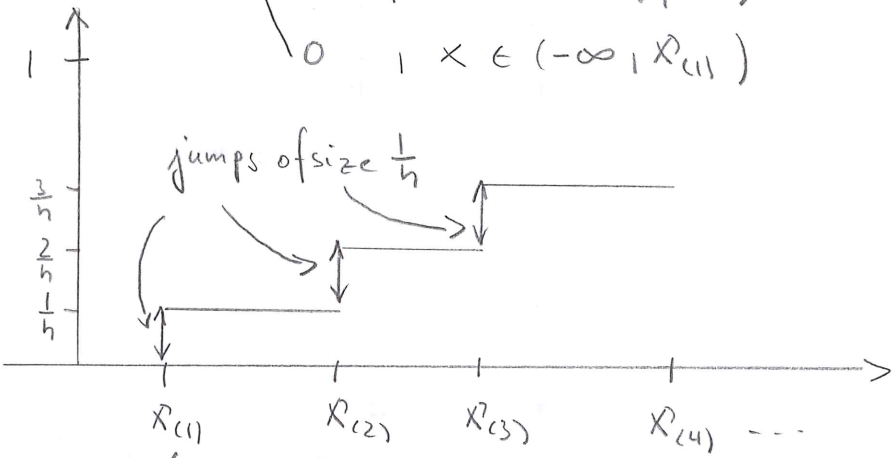
$$X_{(1)} < \ldots < X_{(n)}$$

with prob. 1 (e.g. if X_i i.i.d. with common density f)

(53)

$\overset{\text{def. of } F_n}{\Longrightarrow}$

$$F_n(x) = \begin{cases} \frac{k}{n}, & x \in [X_{(k)}, X_{(k+1)}), \ k = 1, \dots, n-1 \\ 1, & x \in [X_{(n)}, \infty) \\ 0, & x \in (-\infty, X_{(1)}) \end{cases} \qquad (10.3.1.1)$$

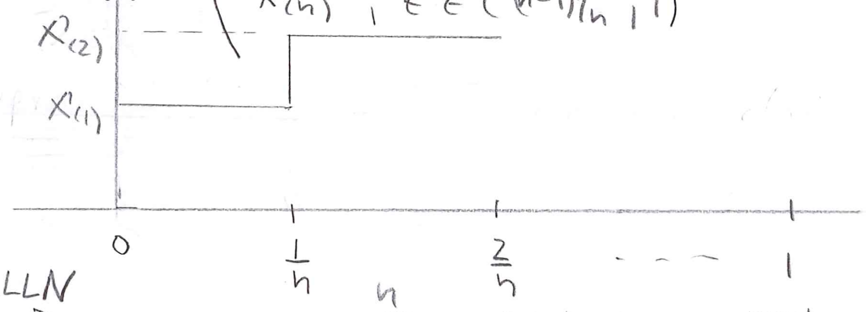


Since F_n is a distr. function we can def. the quantile function $F_n^{\leftarrow}$ of F_n

$\longrightarrow$ $F_n^{\leftarrow}$ <u>empirical quantile function</u>

$\overset{\text{def. of } F^{\leftarrow}}{\Longrightarrow}$ (10.3.1.1)

$$F_n^{\leftarrow}(t) = \begin{cases} X_{(k)}, & t \in ((k-1)/n, k/n], \ k = 1, \dots, n-1 \\ X_{(n)}, & t \in ((n-1)/n, 1) \end{cases} \qquad (10.3.1.2)$$



X_i i.i.d $\overset{\text{SLLN}}{\Longrightarrow}$ $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty, x]}(X_i) \xrightarrow[n\to\infty]{} \underbrace{E[1_{(-\infty,x]}(X_1)]}_{= P(X_1 \le x) = F(x)}$

The convergence is even <u>uniform</u> w.r.t. x!

<u>Glivenko-Cantelli Lemma</u>:

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \xrightarrow[n\to\infty]{} 0$$

$\Longrightarrow$ F_n <u>approximation</u> of the unknown claim size distr. F for large n

The Glivenko-Cantelli Lemma also implies that

$$F_n^{\leftarrow}(t) \xrightarrow[n\to\infty]{} F^{\leftarrow}(t)$$

for all continuity points t of $F^{\leftarrow}$

$\Longrightarrow$ $F_n^{\leftarrow} \approx F^{\leftarrow}$ for large n

$\longrightarrow$ <u>Conclusion</u>: If F is a realistic model for claim size distr. <u>then</u> the graph of

$$(F_n^{\leftarrow}(t), F^{\leftarrow}(t)), \ 0 < t < 1$$

$\overset{(10.3.1.2)}{}$ will (roughly) look like a straight line for large n

$\underset{t=\frac{k}{n+1}}{\Longrightarrow}$ The graph of

$$\left(R_{(k)}, F^{\leftarrow}\left(\tfrac{k}{n+1}\right)\right), k=1,\dots,n \qquad (10.3.1.3)$$

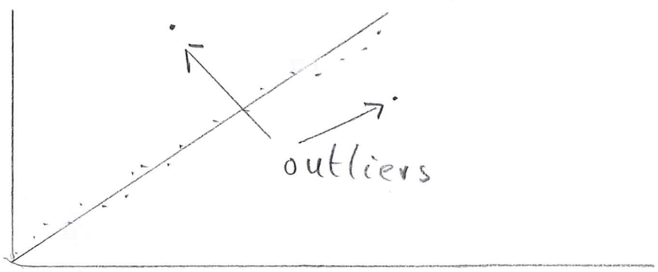
will (roughly) be a straight line

⟶ The graph of (10.3.1.3) is called <u>QQ-plot</u>

⟶ QQ-plot measure for the goodness of fit of claim size data to the model

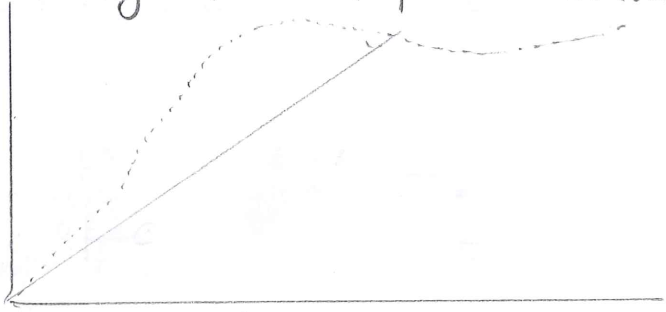
<u>Properties of QQ-plots</u>:

(i) <u>Outliers</u>:



⟶ F still a good model for the claim size distr.

(ii) <u>Shape</u>:



⟶ graph curves down at the right

⟹ QQ-plot indicates that the right tail of the real distr. of claim size data is heavier tailed than in the model (i.e. F)

<u>10.3.2 Mean excess plots</u>

Which criteria are important for a realistic modelling of claim size distributions?

⟶ <u>crucial criterion</u>: F <u>models</u> the probability <u>for large values of claim sizes</u>, i.e. the <u>tails</u> of the (real) claim size distr. appropriately

From the viewpoint of insurance modelling of <u>heavy-tailed claim</u> size distr. is an important issue, since higher probabilities for large claim size values in a portfolio may ruin insurance companies

55 One tool to discriminate heavy-tailed distr. from light-tailed ones is given by

Def. 10.3.2.1 (Mean excess function)

Let $Y \geq 0$ with distr. F s.t. $E[Y] < \infty$

Set $x_l := \inf\{x : F(x) > 0\}$ and $x_r := \sup\{x : F(x) < 1\}$

($\Rightarrow$ $F(x) = 0$, $x < x_l$ and $F(x) = 1$, $x \geq x_r$)

Then the function

$$e_F(u) := E[Y - u \mid Y > u], \quad u \in (x_l, x_r)$$

is called mean excess function or mean excess loss function

$\rightarrow$ expected claim size excess over the threshold value u

$\rightarrow$ link between e_F and the tail distribution $\bar{F} = 1 - F$:

$$e_F(u) = \frac{1}{\bar{F}(u)} \int_u^\infty \bar{F}(y)\, dy, \quad u \in [0, x_r) \qquad (10.3.2.1)$$

and

$$\bar{F}(x) = \frac{e_F(0)}{e_F(x)} \exp\left\{ - \int_0^x \frac{1}{e_F(y)}\, dy \right\}, \qquad (10.3.2.2)$$

if F continuous and $F(x) > 0$, $x > 0$

The def. of e_F suggests to call F

(i) heavy-tailed, if $e_F(u) \xrightarrow[u \to \infty]{} \infty$

and (ii) light-tailed, if $e_F(u) \xrightarrow[u \to \infty]{} \alpha < \infty$

Ex. 10.3.2.2 ($Y \sim Exp(\lambda)$)

$$e_F(u) = \lambda^{-1}, \quad u > 0 \quad \Rightarrow \quad F \text{ light-tailed}$$

Ex. 10.3.2.3 (Y Pareto distr.)

$$\bar{F}(x) = 1 - F(x) = \left(\frac{\kappa}{\kappa + x}\right)^\alpha, \quad \alpha, \kappa > 0$$

$\rightarrow$ $e_F(u) = \dfrac{\kappa + u}{\alpha - 1}$, $\alpha > 1$ $\Rightarrow$ $\lim_{u \to \infty} e_F(u) = \infty$

$\Rightarrow$ F heavy-tailed

Def. 10.3.2.4 (Empirical mean excess function)

Let $F_n(x) = \frac{1}{n} \sum_{i=1}^n 1_{(-\infty, x]}(X_i)$ the emp. distr. function

Then the mean excess function of F_n, i.e. e_{F_n} is called empirical mean excess function

$e_{F_n} \approx e_F$ for large n follows from

Prop. 10.3.2.5: Let X_i be i.i.d. claim sizes with

with distr. F s.t. $E[X_1] < \infty$. Further assume that $F(x) < 1$ for all x (i.e. $x_r = \infty$).

Then
$$e_{F_n}(u) \xrightarrow[n\to\infty]{} e_F(u)$$
with probability 1 for all $u > 0$

Proof:
$$e_{F_n}(u) = \frac{1}{\underbrace{P(Y>u)}_{=\bar{F}_n(u)}} E[\underbrace{1_{\{Y>u\}}(Y-u)}_{=(Y-u)_+ = \max(Y-u,0)}], \quad Y \sim F_n$$

$$\overset{Ex.}{=} \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i-u)_+}{\bar{F}_n(u)} \xrightarrow{SLLN} \frac{E[(X_1-u)_+]}{\bar{F}(u)} = e_F(u)$$

$\frac{1}{n}\sum_{i=1}^{n}(X_i-u)_+ \to E[(X_1-u)_+]$, $\bar{F}_n(u) \xrightarrow{SLLN} \bar{F}(u)$

$\Rightarrow$ proof

$\longrightarrow$ The graph of
$$(u, e_F(u)), \quad u > 0$$
is "approximately" the graph of
$$(X_{(k)}, e_{F_n}(X_{(k)})), \quad k = 1, \dots, n-1 \qquad (10.3.2.3)$$

$\longrightarrow$ The graph (10.3.2.3) is called <u>mean excess plot (ME-plot)</u>

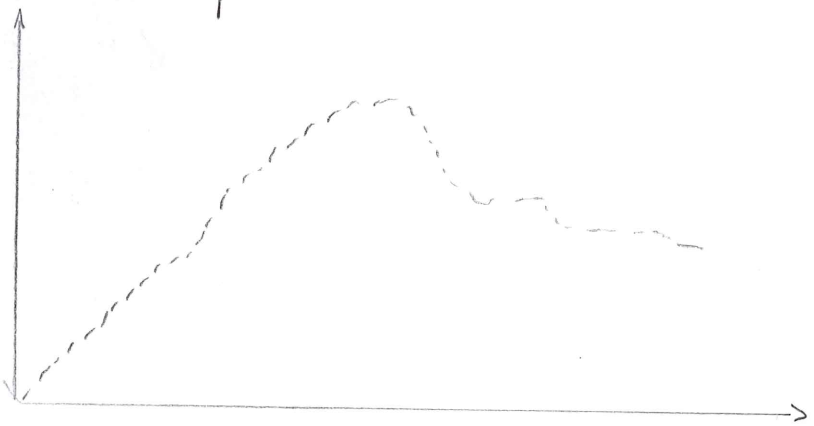
def. of e_F $\longrightarrow$ <u>interpretation</u>:

(i) graph of the ME-plot curves down at the right
$\longrightarrow$ indication that the real claim size distr. is light-tailed

(ii) graph of the ME-plot curves up at the right
$\longrightarrow$ indication for a heavy-tailed claim size distr.

<u>disadvantage</u>: This graphical method to discriminate light-tailed distr. from heavy-tailed is not very robust $\longrightarrow$ better method: e.g. median excess plots

<u>Example</u>: ME-plot



$\longrightarrow$ graph of the ME-plot curves down at the right end

$\longrightarrow$ The ME-plot indicates that the right tail of the (real) claim size distr. is not too dangerous i.e. light-tailed

(57)

10.4. Distribution of the total claim amount $S(t)$

Recall that

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

under our standard assumption: The claim numbers $N(t)$ are independent of the i.i.d claim sizes $X_i \geq 0$, $i \geq 1$

→ Objective: We want to study the distr. of $S(t)$.

Or more generally:

Consider a decomposition

$$S(t) = S_1(t) + \dots + S_n(t)$$

of $S(t)$ into total claim amounts $S_i(t)$, $i=1,\dots,n$ over certain periods of time or with respect to certain claim size layers

→ What can we say about the distributional properties of the sub-portfolios $S_i(t)$, $i=1,\dots,n$?

A useful tool for characterizing the distr. of $S(t)$ are

10.4.1 Mixture distributions

We need the following fundamental concept from probability theory:

Def. 10.4.1.1 (Characteristic function)

Let X be a r.v. Then the function

$$\phi_X(\lambda) := E[e^{i\lambda X}] := E[\cos(\lambda \cdot X)] + i\, E[\sin(\lambda \cdot X)],$$

$\lambda \in \mathbb{R}$

is called characteristic function of X

$(i^2 = -1, \; |e^{i\lambda \cdot x}| = 1)$

Def. 10.4.1.2 (Multidim. characteristic function)

Let $(X_1, \dots, X_n)$ be a random vector. Then the function

$$\phi_{X_1,\dots,X_n}(\lambda_1,\dots,\lambda_n) := E[e^{i(\lambda_1 X_1 + \dots + \lambda_n X_n)}]$$

$$:= E[\cos(\lambda_1 X_1 + \dots + \lambda_n X_n)] + i \cdot E[\sin(\lambda_1 X_1 + \dots + \lambda_n X_n)],$$

$\lambda_1, \dots, \lambda_n \in \mathbb{R}$

is the (n-dim.) characteristic function of $(X_1, \dots, X_n)$

Properties of the characteristic function:

(i) Let $(X_1,\dots,X_n)$ and $(Y_1,\dots,Y_n)$ be random vectors. Then

$$(X_1,\dots,X_n) \overset{d}{=} (Y_1,\dots,Y_n)$$

$\iff$

$$\phi_{X_1,\dots,X_n}(\lambda_1,\dots,\lambda_n) = \phi_{Y_1,\dots,Y_n}(\lambda_1,\dots,\lambda_n)$$

for all $\lambda_1,\dots,\lambda_n \in \mathbb{R}$

(ii) $Z_1,\dots,Z_n$ are independent

$\iff$

$$\phi_{Z_1,\dots,Z_n}(\lambda_1,\dots,\lambda_n) = \phi_{Z_1}(\lambda_1)\cdot\ldots\cdot\phi_{Z_n}(\lambda_n)$$

for all $\lambda_1,\dots,\lambda_n \in \mathbb{R}$

Rem. The generating function is a similar concept to $\phi_{X_1,\dots,X_n}$ for discrete r.v.'s

Notation: $S = S(t)$ and $N = N(t)$ for fixed t

Ex. 10.4.1.3 (Computation of ϕ_S)

$$\phi_S(\lambda) \overset{\text{def.}}{=} E[e^{i\lambda(\sum_{i=1}^{N} X_i)}] = E[e^{i\lambda(\sum_{n\geq 0}(\sum_{i=1}^{n} X_i) 1_{\{N=n\}})}]$$

$$= E[\sum_{n\geq 0} (e^{i\lambda(\sum_{i=1}^{n} X_i)}) 1_{\{N=n\}}] = \sum_{n\geq 0} E[(e^{i\lambda(\sum_{i=1}^{n} X_i)}) \cdot 1_{\{N=n\}}]$$

indep. r.v.'s

$$= \sum_{n\geq 0} \underbrace{E[e^{i\lambda(\sum_{i=1}^{n} X_i)}]}_{=\phi_{X_1,\dots,X_n}(\lambda_1,\dots,\lambda)} \cdot \underbrace{E[1_{\{N=n\}}]}_{=P(N=n)}$$

$$\overset{\text{property (ii) of } \phi}{=} \sum_{n\geq 0} P(N=n)\left(\prod_{i=1}^{n} \phi_{X_i}(\lambda)\right) \overset{\text{property (i) of } \phi}{=} \sum_{n\geq 0} P(N=n)(\phi_{X_1}(\lambda))^n \qquad (10.4.1.1)$$

Ex 10.4.1.4 (ϕ of a compound Poisson sum)

$$N \sim \text{Pois}(\lambda) \iff P(N=n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

$$\overset{(10.4.1.1)}{\Longrightarrow} \phi_S(\tilde\lambda) = \sum_{n\geq 0} \frac{(\lambda\cdot\phi_{X_1}(\tilde\lambda))^n}{n!} e^{-\lambda} = e^{-\lambda(1-\phi_{X_1}(\tilde\lambda))}$$

If e.g. $X_1 \sim U(0,1) \Rightarrow \phi_{X_1}(\lambda) = \int_0^1 e^{i\lambda u} du = \frac{1}{i\lambda} e^{i\lambda u}\Big|_{u=0}^{1}$

Def. 10.4.1.5 (Mixture distribution)

Let $p_i, i=1,\dots,n$ be probabilities for $1,\dots,n$ (i.e. $\sum_{i=1}^{n} p_i = 1$) and $F_i, i=1,\dots,n$ be distr. functions of r.v.'s. Then the distr.

$$G(x) := p_1 F_1(x) + \dots + p_n F_n(x)$$

is called mixture distribution of $F_1,\dots,F_n$

Rem. 10.4.1.6: Let J be a r.v. s.t. $P(J=i) = p_i$, $i=1,\dots,n$. Assume that J is indep. of r.v's $Y_i, i=1,\dots,n$ with $F_{Y_i} = F_i$.

59 Then one shows for the r.v.

$$Z := 1_{\{J=1\}} \cdot Y_1 + \dots + 1_{\{J=n\}} Y_n$$

that $F_Z(x) = G(x)$ and $\phi_Z(x) = p_1 \phi_{Y_1}(x) + \dots + p_n \phi_{Y_n}(x)$

Use the method in Ex. 10.4.1.3

The following result has interesting applications in insurance

Theorem 10.4.1.7 (Sums of independent compound Poisson r.v.'s are compound Poisson)

Let

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)}$$

be independent compound Poisson sums, where $N_i \sim Pois(\lambda_i)$, $\lambda_i > 0$, and $(X_j^{(i)})_{j \geq 1}$ are i.i.d claim sizes for each i.

Then

$$\tilde{S} = S_1 + \dots + S_n$$

is again compound Poisson with

$$\tilde{S} \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i, \quad N_\lambda \sim Pois(\lambda), \lambda = \lambda_1 + \dots + \lambda_n$$

N_λ indep. of the i.i.d (Y_i) with mixture distr.

$$F_{Y_1}(x) = G(x) = \sum_{i=1}^{n} p_i \cdot F_{X_1^{(i)}}(x), \quad p_i = \lambda_i / \lambda$$

Rem.: Note that $(X_j^{(1)})_{j \geq 1}, \dots, (X_j^{(n)})_{j \geq 1}$ are not assumed to be mutually i.i.d

Proof:

$$\phi_{\tilde{S}}(r) \overset{\text{prop. (ii) of } \phi}{=} \phi_{S_1}(r) \cdot \ldots \cdot \phi_{S_n}(r) \overset{\text{Ex. 10.4.1.4}}{=} \prod_{i=1}^{n} e^{-\lambda_i (1 - \phi_{X_1^{(i)}}(r))} = \exp(-\lambda \sum_{i=1}^{n} p_i (1 - \phi_{X_1^{(i)}}(r)))$$

$$= \exp(-\lambda (1 - \underbrace{\sum_{i=1}^{n} p_i \phi_{X_1^{(i)}}(r)}_{= \phi_Y(r)})) \overset{\text{Rem. 10.4.1.6}}{=} \exp(\lambda(1 - \phi_Y(r))),$$

where $Y = \sum_{i=1}^{n} 1_{\{J=i\}} \cdot U_i$ with r.v.'s U_i s.t. $F_{U_i} = F_{X_1^{(i)}}$, $i = 1, \dots, n$.

Ex. 10.4.1.4, prop. (i) of ϕ: $\tilde{S} \overset{d}{=} \sum_{i=1}^{N_\lambda} Y_i$ with $F_{Y_1} = F_Y$ $\Rightarrow$ proof

Ex. 10.4.1.8: Let $N(t), t \geq 0$ be claim numbers modelled by a Poisson process with mean value function $\mu(t)$. Further let $(X_j^{(i)})$ be the claim sizes in year $i \in \mathbb{N}$.

Assume that

$(X_j^{(i)})$ is i.i.d

for each $i \in \mathbb{N}$

as well as

$X_j^{(i)}$ are mutually independent and indep. of N