(60) $\longrightarrow$ The total claim amount in year i is given by

$$S_i := \sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)}$$

One shows (Exerc. 9) that

S_i mutually indep.

and

$$\left(\sum_{j=N(i-1)+1}^{N(i)} X_j^{(i)}\right)_{i=1,\dots,n} \stackrel{d}{=} \left(\sum_{j=1}^{N(i-1,i]} X_j^{(i)}\right)_{i=1,\dots,n} \qquad = N(i)-N(i-1)$$

Set $N_i = N(i)-N(i-1)$. Then

$$S_i \stackrel{d}{=} \sum_{j=1}^{N_i} X_j^{(i)},$$

where

$$N_i \sim \mathrm{Pois}(\mu(i-1,i])$$

$\Longrightarrow$ S_i indep. compound Poisson sums

Denote by $S(n)$ the total claim amount up to n years, that is

$$S(n) = S_1 + \dots + S_n$$

Th. 10.4.1.7 $\Longrightarrow$

$$S(n) \stackrel{d}{=} \sum_{i=1}^{N_\lambda} Y_i,$$

where

$$N_\lambda \sim \mathrm{Pois}(\lambda), \quad \lambda = \mu(0,1] + \dots + \mu(n-1,n] = \mu(n).$$

The r.v.'s Y_i are i.i.d with

$$Y_i \stackrel{d}{=} 1_{\{\eta=1\}} \cdot X_1^{(1)} + \dots + 1_{\{\eta=n\}} X_1^{(n)},$$

where the r.v. $\eta \in \{1,\dots,n\}$ is indep. of $X_1^{(1)},\dots,X_1^{(n)}$.

The distr. of η is given by

$$P(\eta=i) = \mu(i-1,i]/\lambda$$

$\Longrightarrow$ $S(n)$ is compound Poisson again with i.i.d claim sizes Y_i!

<u>Note:</u> It was <u>not</u> assumed that

$X_{j_1}^{(i_1)}$ has the same distr. as $X_{j_2}^{(i_2)}$

for $i_1 \neq i_2$

Ex. 10.4.1.9 : Consider n indep. portfolios with total claim amounts

$$S_i = \sum_{j=1}^{N_i} X_j^{(i)},$$

where

$$N_i \sim \mathrm{Pois}(\lambda_i)$$

and

$$N_i \text{ indep. of i.i.d } (X_j^{(i)})$$

for each i.

For example S_i is the total claim amount in a portfolio of car insurance policies within the tariff group (or account) i

Th 10.4.1.7 ⟹ The aggregation of $S_i, i=1,\dots,n$, that is

$$S = S_1 + \dots + S_n$$

is compound Poisson again with counting variable

$$N_\lambda \sim \mathrm{Pois}(\lambda), \quad \lambda = \lambda_1 + \dots + \lambda_n$$

10.4.2 Space-time decomposition of the total claim amount

Now let us consider the converse problem to Th. 10.4.1.7, that is we want to study sub-portfolios of a given portfolio

More precisely, consider the total claim amount

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i\geq 1} X_i 1_{[0,t]}(T_i)$$

where N is a Poisson proc. with mean value function $\mu(t)$ indep. of the i.i.d claim sizes (X_i).

T_i are the claim arrivals.

Set

$$E = \underset{\text{time}}{[0,\infty)} \times \underset{\text{claim size values}}{[0,\infty)}$$

Let $A_1, \dots, A_n$ be a disjoint partition of E, that is

$$\bigcup_{i=1}^{n} A_i = E, \quad A_i \cap A_j = \emptyset, \quad 1 \leq i < j \leq n$$

⟶ decomposition of $S(t)$ into total claim amounts of sub-portfolios over certain periods of time and w.r.t. certain claim size layers:

$$S(t) = S_1(t) + \dots + S_n(t),$$

where
$$S_j(t) = \sum_{i=1}^{N(t)} X_i \cdot 1_{A_j}((T_i, X_i)), \quad j = 1, \dots, n+1$$

Ex. 10.4.2.1 (Partitioning in time)
Choose $0 = t_0 < t_1 < \dots < t_n = t$ and set
$$\Delta_1 = [0, t_1], \quad \Delta_i = (t_{i-1}, t_i], \ i = 2, \dots, n, \quad \Delta_{n+1} = (t_n, \infty)$$
$\longrightarrow$ $A_i := \Delta_i \times [0, \infty)$, $i = 1, \dots, n+1$

Ex. 10.4.2.2 (Partitioning in the claim size space)
Consider the different layers of claim sizes
$$B_1 = [0, d_1], \ B_2 = (d_1, d_2], \dots, B_n = (d_{n-1}, d_n], \ B_{n+1} = (d_n, \infty)$$
for $0 < d_1 < \dots < d_n < \infty$
$\longrightarrow$ $A_i := [0, t] \times B_i$, $i = 1, \dots, n+1$
For example different insurance companies share the risk of a portfolio by each managing sub-portfolios $S_i(t)$ w.r.t. distinct claim size magnitudes $X_j^{(i)} \in B_i$
$\longrightarrow$ converse result to Th. 10.4.1.7

Theorem 10.4.2.3 (Space-time decomposition of $S(t)$)
Let N be a Poisson proc with $\mu(t) = \int \lambda(s)\,ds$, where $\lambda > 0$ is a continuous intensity function. Further let $A_1, \dots, A_n$ be a disjoint partition of $E = [0,\infty) \times [0,\infty)$. Then
$$S_j(t) = \sum_{i=1}^{N(t)} X_i 1_{A_j}((T_i, X_i)), \ j = 1, \dots, n$$
are independent for all $t \geq 0$. Moreover each $S_j(t)$ is compound Poisson with
$$S_j(t) \stackrel{d}{=} \sum_{i=1}^{N(t)} X_i 1_{A_j}((Y_i, X_i)),$$
where (Y_i) are i.i.d r.v.'s with density $\lambda(x)/\mu(t)$, $0 \leq x \leq t$, independent of N and (X_i)

Proof: More or less a direct consequence of the order statistics property of the Poisson process (Th. 9.1.4.1) see Mikosch.

10.4.3 Panjer recursion scheme

$\longrightarrow$ Exact numerical procedure for calculating the distr. of the total claim amount
$$S(t) = \sum_{i=1}^{N(t)} X_i$$

Notation: $S = S(t)$ and $N = N(t)$ for fixed t
$$S_n := \sum_{i=1}^{n} X_i$$

$\longrightarrow$ Theorem 10.4.3.1 (Panjer recursion)

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Assume that the claim sizes X_i take values in $\{0,1,2,\dots\}$ and that the claim number N has distr. given by

$$q_n := P(N=n) = \left(a+\frac{b}{n}\right) q_{n-1} \quad , n=1,2,\dots \quad (10.4.3.1)$$

for some $a,b \in \mathbb{R}$.

Then the probabilities $p_n := P(S=n)$ can be calculated by

$$p_0 = \begin{cases} q_0 & , \text{if } P(X_1=0)=0 \\ E\left[(P(X_1=0))^N\right] & \text{else} \end{cases}$$

$$p_n = \frac{1}{1-a\,P(X_1=0)} \sum_{i=1}^{n} \left(a+\frac{b\cdot i}{n}\right) P(X_1=i)\, p_{n-i} \quad , n \geq 1$$

Rem. 10.4.3.2: Condition (10.4.3.1) is called (a,b)-condition and one shows that this cond. is only satisfied by the following 3 distributions:

1. Pois(λ) $\longrightarrow$ $a=0$, $b=\lambda>0$
2. Binomial distribution $B(n,p)$ $\longrightarrow$ $a=-p/(1-p)<0$, $b=-a(n+1)$, $n\in\mathbb{N}_0$
3. Negative binomial distribution with param. (p,v), that is
$$P(Z=k) = \binom{v+k-1}{k} p^v (1-p)^k \quad , k\in\mathbb{N}_0, \; p\in(0,1), \; v>0$$
$\longrightarrow$ $0<a=1-p<1$, $b=(1-p)(v-1)$, $a+b>0$
See Ex. 9.3.2.

Rem. 10.4.3.3:

(i) The case $X_i \in d\,\mathbb{N}_0 = \{d\cdot n : n\geq 0\}$, $d>0$ can be reduced to the case $X_i\in\mathbb{N}_0$ by rewriting $S = d\sum_{i=1}^{N} \underbrace{(X_i/d)}_{\in\mathbb{N}_0}$

(ii) Every total claim amount with continuous claim size distr. can be approximated by one with $X_i \in d\,\mathbb{N}_0$

Proof of Th. 10.3.4.1: Proof based on the (a,b)-cond.

$$p_0 = P(N=0) + P(S=0, N>0)$$

If $P(X_1=0)=0 \Longrightarrow p_0 = P(N=0) \overset{\text{def}}{=} q_0$

If $P(X_1=0)\neq 0 \Longrightarrow$