Problems and Methods in Actuarial Science (STK 3505)
Obligatorisk oppgave, 23.10.2015

At least 2 problems are supposed to be solved.

Deadline: Thursday, 12.November, 14:30, 7th floor (ekspedisjon, Niels Henrik Abels hus)

Problem 1 Consider the following Danish fire insurance data collected at Copenhagen Reinsurance in the period from 09/30/1988 until 11/07/1988:

Date	Loss X_i in DKM	Date	Loss X_i in DKM
10/01/1988	3.283052	10/23/1988	1.597161
10/04/1988	25.953860	10/28/1988	1.774623
10/04/1988	1.064774	10/29/1988	1.721384
10/08/1988	4.081633	11/02/1988	2.333629
10/08/1988	2.288376	11/03/1988	4.494232
10/17/1988	1.100266	11/05/1988	8.873114
10/20/1988	2.838509	11/05/1988	2.676131
10/20/1988	3.194321	11/07/1988	1.147294

The claim sizes X_i in the above table are expressed in millions of Danish Krone. Assume that the inter-arrival times W_i are described by the Cramer-Lundberg model.

(i) Compute the maximum-likelihood estimator $\widehat{\lambda}$ of the jump intensity of the claim number process $N(t)$.

(ii) Calculate

$$P(W_1 > 5).$$

(iii) Estimate $E[X_1]$ by using the sample mean and calculate the premium $p_{EV}(t)$ with respect to the total claim amount of fire losses given by

$$(1+\rho)E[S(t)]$$

for the safety loading $\rho = 0.20$ and $t = 1$ (year).

Hint: Apply the following convention: Claims $X_1^{(m)}, ..., X_n^{(m)}$ arriving at day $m \geq 0$ are interpreted as 1 claim $X_m := \sum_{i=1}^{n} X_i^{(m)}$ arriving at day m. See also Section 3.1.1 in the book of T. Mikosch.

Problem 2 (i) Given the survival probability

$${}_tp_x = \left(1 - \frac{t}{120 - x}\right)^{1/6}$$

for $0 \le t < 120 - x$. Calculate the expected future life time $\mathring{e}_x = E[T]$ of a life aged $x = 30$ years.

(ii) Consider the following table of one-year death probabilities q_x :

Age x	$1000 \cdot q_x$	Age x	$1000 \cdot q_x$
30	1.53	35	2.01
31	1.61	36	2.14
32	1.70	37	2.28
33	1.79	38	2.43
34	1.90	39	2.60

Assume that the expected number of completed future years lived by a life aged $x = 30$ years is $e_x = 49.5$. Compute e_{x+10}.

Problem 3 Consider the death probabilities (DAV-life table 1994 T for women, Germany):

Age x	$1000 \cdot q_x$	Age x	$1000 \cdot q_x$
30	0.689	35	0.971
31	0.735	36	1.057
32	0.783	37	1.156
33	0.833	38	1.267
34	0.897	39	1.390

The following insurance policy is issued to a woman aged $x = 30$ years: Annual premiums Π_k, $k = 0, ..., 4$ are payable by the insured until death occurs within the contract period of $n = 5$ years. The insurer provides 15000 at the end of the contract period if the insured is still alive after 5 years. Otherwise the insurance provides for a refund of all net premiums paid accumulated at the interest rate $i = 3\%$ at the end of the year of death. Suppose that

$$\Pi_k = (1.03)^k \Pi, \ \ k = 0, ..., 4.$$

Calculate the net premiums $\Pi_k, k = 0, ..., 4$.

Problem 4 Suppose that the future life time T of a life (25) is exponentially distributed with parameter $\lambda > 0$, that is T has the probability density

$$g(t) = \lambda e^{-\lambda t}, \ \ t \ge 0.$$

Assume that $\lambda = 0.0188$.

(i) Determine $\mathring{e}_x = E[T]$.

(ii) Calculate the force of mortality μ_{x+t}.

(iii) Compute the expected remaining life time $E[S]$ in the year of death. Assume here for convenience that the curtate K and S are independent.

Problem 5 Let $i = 5\%$ be the technical interest rate and $\ddot{a}_x = 18.589547$ the net single premium of a life aged $x = 25$ years. Consider following life table data:

Age x	$1000 \cdot q_x$	Age x	$1000 \cdot q_x$
25	1.22	30	1.53
26	1.27	31	1.61
27	1.33	32	1.70
28	1.39	33	1.79
29	1.46	34	1.90

Let us have a look at the following policy with a premium refund: A whole life insurance

issued to a life aged $x = 25$ years provides a payment of $15,000$. Annual premiums are paid at the beginning of the year for the time span of 10 years. Death claims are paid at the end of the year of death. A premium refund feature is in effect during the premium payment period which provides that one half of the last premium paid to the company is refunded as an additional death benefit.

Calculate the net premium reserves ${}_kV, k = 0, ..., 5$ for this insurance policy.

Problem 6 An endowment is issued to a life aged 42 with 67 years as age at maturity. Let $i = 3\%$ be the technical interest rate and assume that the death benefit is 200000\$. Suppose the survival benefit is given by 100000\$.

Further, assume that the mortality rates are described by the generalized Gompertz-Makeham law as follows:

$$\mu_x = \exp(-7.85785 + 0.01538 \cdot x + 5.77355 \cdot 10^{-4} \cdot x^2).$$

Determine the (constant) annual premiums P based on the equivalence principle and compute the net premium reserves.

Problem 7 Let us cast a glance at the Danish fire insurance data of Problem 1.

Now suppose that the total claim amount $S(t)$ of fire losses (in millions of Danish Krone) is described by the Crámer-Lundberg model. Further it is assumed that the law F of the claim sizes X_i is approximated by the empirical distribution function

$$F_n(x) := \frac{1}{n} \sum_{i=1}^{n} 1_{(-\infty,x]}(X_i)$$

for the sample size n of Problem 1.

(i) Compute the maximum likelihood estimator of the intensity λ of fire losses.

(ii) Write a programme to simulate 40 paths of the total claim amount for the time period of 1 year.

Hint: Note that

$$S(t) = \sum_{i=1}^{N(t)} X_i = \sum_{i \geq 1} X_i 1_{[0,t]}(T_i),$$

where $T_i = W_1 + ... + W_i$ are the claim times and W_j the inter-arrival times.

Use existing software packages to simulate the i.i.d sequences of random variables (X_i) and (W_i). Otherwise you can proceed as follows:

Simulation of the uniform law on [0,1]: Generate a sequence of integers $(x_n)_{n\geq 1}$ between 0 and $m-1$ by

$$\begin{cases} x_0 = \text{intitial value} \in \{0, 1, ..., m-1\} \\ x_{n+1} = (a \cdot x_n + b) \bmod m, \end{cases}$$

where

$$\begin{cases} a = 31415821 \\ b = 1 \\ m = 10^8 \end{cases}$$

The numbers $U(\omega_0) = \frac{x_0}{m}, U(\omega_1) = \frac{x_1}{m}, U(\omega_2) = \frac{x_2}{m}, ...$ give the outcomes of a $U \sim U(0,1)$.

Simulation of an exponential distribution:

$$W = -\frac{\log(U)}{\lambda}$$

where U is uniformly distributed on $[0,1]$. So for i.i.d $U_i \sim U(0,1)$ we obtain i.i.d $W_i = \frac{\log(U_i)}{\lambda} \sim Exp(\lambda)$.

Simulation of the claim sizes with law F_n: Consider the quantile function $F_n^{\leftarrow}$ of the empirical distribution function F_n given by

$$F_n^{\leftarrow}(x) := \inf\{x \in \mathbb{R} : F_n(x) \geq t\}, 0 < t < 1.$$

Visualize the average numbers of claims in $(-\infty, x]$, that is $F_n(x)$ for the sample size n to see how the graph of $F_n^{\leftarrow}(x)$ looks like.

Then for i.i.d $U_i \sim U(0,1)$ the sequence $X_i := F_n^{\leftarrow}(U_i)$ becomes i.i.d with common law F_n (as an approximation of F).

Further note that in virtue of the SLLN

$$T_i \longrightarrow \infty \text{ for } i \longrightarrow \infty$$

with probability 1 holds. So $T_i \geq 1$ year for large i with probability 1.