UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Answers

Day of examination: ??. ??. ????

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Grade scale:
A: 62 – 68
B: 52 – 61
C: 39 – 51
D: 31 – 38
E: 27 – 30
F: 0 – 26

Problem 1 General insurance

1a

The moment estimate is given by

$$\overline{median} = \hat{\mu}\log(2)$$

so that

$$\mu = \frac{3.8216}{\log(2)} = 5.5134$$

Theoretical moment should match observed moment.

1b

$$\mathcal{L}(\mu) = \log\left(\prod_{i=1}^{n} f(z_i|\mu)\right) = \sum_{i=1}^{n} \log\left(f(z_i|\mu)\right) = \sum_{i=1}^{n} \log\left(\frac{\exp(-\frac{z_i}{\mu})}{\mu}\right) =$$

$$= \sum_{i=1}^{n} \log\left(\exp(-\frac{z_i}{\mu})\right) - \sum_{i=1}^{n} \log(\mu) = -n\log(\mu) - \sum_{i=1}^{n} \frac{z_i}{\mu}$$

$$\frac{\partial \mathcal{L}(\mu)}{\partial \mu} = -\frac{n}{\mu} + \frac{\sum_{i=1}^{n} z_i}{\mu^2}$$

$-\frac{n}{\mu} + \frac{\sum_{i=1}^{n} z_i}{\mu^2} = 0$ when $\mu = \frac{\sum_{i=1}^{n} z_i}{n}$.

(Continued on page 2.)

1c

Simulation of Z using the inversion method requires an expression for the inverse cdf $F^{-1}(u)$:

$$F(z) = 1 - \exp(-\frac{x}{\mu}) = u$$

$$z = -\mu \log(1-u) = F^{-1}(u).$$

Simulation algorithm:

1. Input: α, β
2. Draw $U^* \sim U(0,1)$
3. Return $Z^* = -\mu \log(1-U^*)$ or $Z^* = -\mu \log(U^*)$

1d

Simulation algorithm for $\mathcal{X}$:

1. Input $\alpha, \beta, \lambda, m$
2. for $i = 1, \ldots, m$ do
3. Draw $\mathcal{N}^* \sim Poisson(\lambda)$
4. $\mathcal{X}_i^* \leftarrow 0$
5. for $j = 1, \ldots, \mathcal{N}^*$ do:
6. Draw $X^* \sim exponential(\mu)$
7. $\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$
8. end for
9. end for
10. Return $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$.

Estimate of the mean: $\overline{\mathcal{X}^*} = \frac{1}{m}\sum_{i=1}^m \mathcal{X}_i^*$.

Estimate of the standard deviation: $S_{\mathcal{X}^*}^* = \sqrt{\frac{1}{m-1}\sum_{i=1}^m (\mathcal{X}_i^* - \overline{\mathcal{X}^*})^2}$

The $100 * \epsilon\%$ reserve r_ϵ is given by $P(\mathcal{X} \leq q_\epsilon) = \epsilon$ and estimated by $q_\epsilon^* = \mathcal{X}_{(m\epsilon)}^*$, where $\mathcal{X}_{(1)}^* \leq \cdots \leq \mathcal{X}_{(m)}^*$.

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1e

$$E(\mathcal{X}^*|\mathcal{N}) = E(\sum_{i=1}^{\mathcal{N}} Z_i) = \sum_{i=1}^{\mathcal{N}} E(Z_i) = \sum_{i=1}^{\mathcal{N}} \mu = \mathcal{N}\mu$$

$$E(\mathcal{X}^*) = E(E(\mathcal{X}^*|\mathcal{N})) = E(\mathcal{N}\mu) = \lambda\mu = 110.2681$$

$$var(\mathcal{X}^*|\mathcal{N}) = var(\sum_{i=1}^{\mathcal{N}} Z_i) = \sum_{i=1}^{\mathcal{N}} var(Z_i) = \mathcal{N}\mu^2$$

$$var(\mathcal{X}^*) = var(E(\mathcal{X}^*|\mathcal{N})) + E(var(\mathcal{X}^*|\mathcal{N})) = var(\mathcal{N}\mu) + E\left(\mathcal{N}\mu^2\right) =$$

$$= \lambda\mu^2 + \lambda\mu^2 = 1215.905$$

$$sd(\mathcal{X}^*) = 34.86983$$

The theoretical and simulated results are consistent.

1f

The 95% and 99% reserves are 172.05 and 203.62.

1g

Pure premium is 110.17 so premium is $110.17 * (1 + 0.3) = 143.221$

1h

$$\mathcal{X}^{re} = \sum Z^{re} = \sum aZ_i = a\sum Z_i = a\mathcal{X}$$

1i

$$\mathcal{X}^{ce} = (1 - 0.4)\mathcal{X}$$

For 95%:

$$0.95 = P(\mathcal{X} < 172.05) = P(\frac{\mathcal{X}^{ce}}{0.6} < 172.05) = P(\mathcal{X}^{ce} < 0.6 \cdot 172.05) = P(\mathcal{X}^{ce}103.23)$$

Similar for 99%.

The 95% reserve is 103.23, the 99% reserve is 122.172

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Problem 2 Life insurance

2a

Probability of surviving from age 60 to age 65 is equal to the probability of surviving 5 years given that the individual is of age 60

$$_5p_{60} = {_1p_{60}} \cdot {_1p_{61}} \cdot {_1p_{62}} \cdot {_1p_{63}} \cdot {_1p_{64}}$$

$_1p_{60}$ corresponds to (1−probability of dying between ages 60 and 61) from the life table. That means that

$$_5p_{60} = (1-0.009093)\cdot(1-0.009768)\cdot(1-0.010467)\cdot(1-0.011181)\cdot(1-0.011922) = 0.94865$$

2b

$_0p_{60} = 1$, $_1p_{60} = 0.990907$, $_2p_{60} = 0.9812278$, $_3p_{60} = 0.9709573$, , $_4p_{60} = 0.960101$.

$$-\pi \sum_{k=0}^{l_r-l_0-1} d^k {_kp_{l_0}} = -1000 \sum_{k=0}^{65-60-1} \left(\frac{1}{1+0.05}\right)^k {_kp_{60}} = -1000 \sum_{k=0}^{4} (0.952381)^k {_kp_{60}} =$$

$$= -1000(1 + 0.952381 \cdot 0.990907 + 0.952381^2 \cdot 0.9812278 + 0.952381^3 \cdot 0.9709573 + 0.952381^4 \cdot 0.9$$

$$= -4717.328$$

Probabilities of survival for women are generally higher than for general population and for men - lower than for general population. Because of that the absolute value of the insurance value after contributing stage for women will be higher than for the general population and for men - lower.

2c

It's easier to simulate the numbers of people surviving and subtract from the original numbers at the end. $p_{i,k}$ - probabilities of surving.

There are two options.

Simulation algorithm 1: use probabilities $_2p_{60}$

1. Input: $l_0 = 60, K = 2, J_M, J_W = 500, p_{1,M}, p_{2,M}, p_{1,W}, p_{2,W}$
2. Draw $\mathcal{N}^*_{2,M} \sim Binomial(J_M, p_{1,M} \cdot p_{2,M})$
3. Draw $\mathcal{N}^*_{2,W} \sim Binomial(J_W, p_{1,W} \cdot p_{2,W})$
4. Return $1000 - \mathcal{N}^*_{2,M} - \mathcal{N}^*_{2,W}$

Simulation algorithm 2: use probabilities $_1p_{60}, {_1p_{61}}$

1. Input: $l_0 = 60, K = 2, J_M, J_W = 500, p_{1,M}, p_{2,M}, p_{1,W}, p_{2,W}$
2. Draw $\mathcal{N}^*_{1,M} \sim Binomial(J_M, p_{1,M})$

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3. Draw $\mathcal{N}^*_{1,W} \sim Binomial(J_W, p_{1,W})$
4. Draw $\mathcal{N}^*_{2,M} \sim Binomial(\mathcal{N}^*_{2,M}, p_{2,M})$
5. Draw $\mathcal{N}^*_{2,W} \sim Binomial(\mathcal{N}^*_{2,M}, p_{2,W})$
6. Return $1000 - \mathcal{N}^*_{2,M} - \mathcal{N}^*_{2,W}$

Problem 3 Financial risk

3a

The premium for put option in terms of single asset is

$$\pi(v_0) = e^{-rT} E_Q(max(R - r_g, 0))v_0,$$

3b

The premium for put option in terms of single asset is

$$\pi(v_0) = e^{-rT} E_Q(max(R - r_g, 0))v_0,$$

where $R = e^{\xi_q T + \sigma\sqrt{T}\epsilon} - 1$ for $\epsilon \sim N(0, 1)$.

There is positive payoff if $R > r_g$ or equivalently if $\epsilon > a$ where

$$a = \frac{log(1 + r_g) - \xi_q T}{\sigma\sqrt{T}}$$

and the option premium becomes

$$\pi(v_0) = e^{-rT} \left(\int_a^\infty (e^{\xi_q T + \sigma\sqrt{T}x} - 1 - r_g)\phi(x)dx \right).$$

Splitting the integrand gives

$$\pi(v_0) = e^{-rT} \left(-(1 + r_g) \int_a^\infty \phi(x)dx + e^{\xi_q T} \int_a^\infty e^{\sigma\sqrt{T}x}\phi(x)dx \right)$$

where the second integral on the right is

$$\int_a^\infty e^{\sigma\sqrt{T}x}(2\pi)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}dx = e^{\sigma^2\frac{T}{2}} \int_a^\infty (2\pi)^{-\frac{1}{2}}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}dx.$$

If $\Phi(x) = \int_{-\infty}^x \phi(y)dy$ is the Gaussian integral then $1 - \Phi(x) = \Phi(-x)$ and

$$\pi(v_0) = e^{-rT}(-(1 + r_g)(\Phi(-a)) + e^{\xi_q T + \sigma^2\frac{T}{2}}(\Phi(-a + \sigma\sqrt{T})))v_0$$

and inserting $\xi_q = r - \frac{\sigma^2}{2}$ into the expression and into a yields the Black-Scholes formula.

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3c

1. Input: parameters $r_g, r, \sigma, T, m, w_1, w_2, \rho$ etc
2. for $i = 1, \dots, m$ do:
3. Draw $\epsilon_1, \epsilon_2 \sim N(0,1)$ iid
4. $\epsilon_2 \leftarrow \rho\epsilon_1 + \sqrt{1-\rho^2}\epsilon_2$
5. $R^*_{i,1} \leftarrow e^{rT - \sigma_1^2 \frac{T}{2} + \sigma_1\sqrt{T}\epsilon_1} - 1$
6. $R^*_{i,2} \leftarrow e^{rT - \sigma_2^2 \frac{T}{2} + \sigma_2\sqrt{T}\epsilon_2} - 1$
7. $X^*_i \leftarrow max(\frac{1}{3}R^*_{i,1} + \frac{2}{3}R^*_{i,2} - r_g, 0)$
8. end for
9. return $\pi^* = \frac{e^{-rT}}{m}\sum_{i=1}^m X^*_i$

the higher the correlation - the higher the price

3d

1. Input: parameters r_g, r, σ, T, m etc
2. for $i = 1, \dots, m$ do:
3. Draw $\epsilon_1, \epsilon_2 \sim N(0,1)$ iid
4. $R^*_{i,1} \leftarrow e^{rT - \sigma_1^2 \frac{T}{2} + \sigma_1\sqrt{T}\epsilon_1} - 1$
5. $R^*_{i,2} \leftarrow e^{rT - \sigma_2^2 \frac{T}{2} + \sigma_2\sqrt{T}\epsilon_2} - 1$
6. $X^*_i \leftarrow max(R^*_{i,1} - R^*_{i,2}, 0)$
7. end for
8. return $\pi^* = \frac{e^{-rT}}{m}\sum_{i=1}^m X^*_i$

END