Note: *You do not have to answer more than four of the five problems below and you choose which ones yourself. Mark A is eminently possible with four out of five! Try to provide answers that are as brief and precise as possible.*

Problem 1

Consider property insurance and let X be the total claims from a single policy holder during one year.

a) What is meant by the pure premium of the policy? Express the definition in mathematical terms.

b) What does premium loading signify? Why must the company charge more than the pure premium?

Suppose a company has taken responsibility for a portfolio of J polices and that the risk they represent is independent of each other. Let $X_1, \dots, X_J$ be the payments to these policy holders during one year.

c) What is the reserve of the portfolio? Express it mathematical terms.

d) Argue that the situation permits full diversification (i.e that relative risk approaches 0 as $J \to \infty$) by showing that if $\xi = E(X_j)$ and $\sigma = E(X_j)$ are common for all policies, then

$$\frac{\mathrm{sd}(X_1 + \ldots + X_J)}{E(X_1 + \ldots + X_J)} \to 0 \qquad \text{as} \qquad J \to \infty.$$

Problem 2

Suppose all individiual claims of the preceding problem have the same probability distribution so that the total claim $\mathcal{X}$ against the portfolio can be written

$$\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$$

where $\mathcal{N}$ is the number of claims. Assume that $\mathcal{N}$ is Poisson distributed with parameter λ and that the individual damages per incident $Z_1, Z_2, \ldots$ are Pareto distributed with density and distribution functions

$$f(z) = \frac{\alpha/\beta}{(1 + z/\beta)^{1+\alpha}} \qquad \text{og} \qquad F(z) = 1 - \frac{1}{(1 + z/\beta)^{\alpha}}$$

where α og β are positive parametres. All of $Z_1, Z_2, \ldots$ are mutually independent among themselves and of $\mathcal{N}$. You may without proof use that $E(Z) = \beta/(\alpha - 1)$.

a) You won't always find Pareto samplers in public software. Derive one yourself through the inversion algorithm which transfers a uniform U^* to a Pareto variable Z^*.

b) Sketch a program which generates m simulations of $\mathcal{X}$.

The table below shows expectation, standard deviation and a few percentiles of the distribution of $\mathcal{X}$ when $\lambda = 10$, $\alpha = 5$ and $\beta = 10$. The number of simulations were 100000:

$E(\mathcal{X})$	sd$(\mathcal{X})$	1%	5%	25%	50%	75%	95%	99%
25.0	12.8	4.3	8.0	15.8	23.0	31.9	48.8	64.2

c) Check the program by comparing $E(\mathcal{X})$ of the table with the exact value you compute from the information above. Is there sign of anything being wrong?

d) What is the 95% og 99% reserve of the portfolio?

Suppose some of the risk is re-insured to a second company which covers the amount exceeding $a = 30$. This means that $\mathcal{X}^{\text{re}} = \max(\mathcal{X} - a, 0)$ is transfered back to the cedent.

e) How is the re-insurer risk simulated?

The table below, computed from 100000 simulations, shows the probability distribution of $\mathcal{X}^{\text{re}}$:

$E(\mathcal{X}^{\text{re}})$	sd$(\mathcal{X}^{\text{re}})$	1%	5%	25%	50%	75%	95%	99%
3.15	7.37	0	0	0	0	1.94	18.76	34.25

f) What is the pure premium of the re-insurance and what is the 95% og 99% reserve of the re-insurer?

Problem 3

Suppose an individual of age l_0 sets up an insurance contract which comprises a pension s from the age l_0+K, lasting to the end of life. Premia π are paid in advance from age l_0 until age l_0+K-1, and the company takes over the capital if the individual dies during this period. Let p_l be the probability of survival to age $l+1$ when age l has been reached and introduce $d = 1/(1+r)$ as the discount factor.

a) Argue that the probability ${}_kp_{l_0}$ for the individual being alive at age $l_0 + k$ is

$${}_kp_{l_0} = p_{l_0}p_{l_0+1}\cdots p_{l_0+k-1}.$$

b) Use these quantities to express the value V_0 of the contact when it is set up. Use sums or mortality adjusted annuities.

c) What is the equivalence premium of the contract?

d) Suppose the policy holder is alive at time k. Define the value of the contact at this point in time through an expression similar to the one in b).

e) Suppose the contract is cancelled after k periods. How much is the policy holder re-paid?

Problem 4

The random walk on logaritmic scale is a common model for equity. This means that if S_k is the value at time k and $Y_k = \log(S_k)$, then S_k evolves according to

$$Y_k = Y_{k-1} + X_k \qquad \text{and} \qquad S_k = e^{Y_k}$$

where $X_1, X_2, \ldots$ are stochastic independent and identically distributed. The recursion starts at $Y_0 = \log(s_0)$ where s_0 is the value of the stock at time 0. Assume that X_k is normal with expectation

ξ og standard deviation σ. You will below need the result that if Z is normal with η og standard deviation τ, then $E(e^Z) = e^{\eta+\tau^2/2}$.

a) First show that the return in period k may be expressed as $R_k = e^{X_k} - 1$.

b) Argue that Y_k under the assumptions above is normally distributed with expectation $\log(s_0) + k\xi$ and standard deviation $\sqrt{k}\,\sigma$ and also that $E(S_k) = s_0 e^{k\xi + k\sigma^2/2}$.

c) Sketch how you simulate $S_1, \ldots, S_K$.

How the stock evolves were simulated 100000 times when $\xi = 0.05$, $\sigma = 0.2$, $s_0 = 1$ and $K = 10$. That lead to the following table of the probability distribution of S_K:

$E(S_K)$	sd(S_K)	1%	5%	25%	50%	75%	95%	99%
2.01	1.41	0.38	0.58	1.07	1.64	2.52	4.67	7.16

d) Does the expectation entry correspond to the theoretical value you may compute from the information above? Brief comment.

e) How can the table be used to approximate the probability distribution of the ten-year returns? For example, how likely is it under this model that money is lost after such a long time? What is the probability that the capital is doubled? What about four-doubled?

Problem 5

The stock investmenet of the preceding problem may be combined with a cash account earning the floating rate of interest. If B_k is the value of the account at time k, then

$$B_k = (1 + r_k)B_{k-1}, \qquad k = 1, 2, \ldots \qquad \text{starting at} \qquad B_0 = b_0.$$

Assume for interest rate the model $r_k = \zeta e^{Z_k}$ where

$$Z_k = aZ_{k-1} + \tau\varepsilon_k, \qquad k = 1, 2, \ldots \qquad \text{starting at} \qquad Z_0 = \log(r_0/\zeta).$$

Here ζ, a og τ are parametres and $\varepsilon_1, \varepsilon_2, \ldots$ independent standard normal variables with expectation 0 og standard deviation 1.

a) Sketch how you simulate $B_1, \ldots, B_K$.

b) How can the program be combined with the program in Problem 4c) so that the total values $V_k = S_k + B_k$ for $k = 1, 2, \ldots$ of cash account and stock *together* are simulated. Assume that stock market and rates of interest vary independently of each other.

Let the capital in the beginning (at time 0) be 1 as in Problem 4, but reduce the stock investment to 0.5 and put the rest in the bank. Assume the model in Problem 4 for equity and suppose in the interest rate model $\zeta = 0.03$, $a = 0.6$, $\tau = 0.25$ og $r_0 = 0.03$. Using 100000 simulations lead to the following probability distribution of V_K after $K = 10$ years:

$E(V_K)$	sd(V_K)	1%	5%	25%	50%	75%	95%	99%
1.69	0.70	0.86	0.97	1.22	1.50	1.94	3.01	4.25

c) How much has the substitution of half the stock investment for cash influenced the probability distribution of the ten-year returns? Compare with the table in Problem 4 through a brief discussion.