Introduction

Problem 1 is on financial risk, Problem 2 on life insurance and Problem 3 on property insurance and all of them must be answered. Write program sketches in any style you find suitable. The mathematical definitons and expressions below which you are **not** to prove or justify in any way, will help you solve some of the sub-problems.

Material to build on

How financial returns accumulate will be central in Problem 1. If $R_1, \ldots, R_K$ are the returns in K consecutive periods, then the return after sitting on the investment until time K is

$$R_{0:K} = (1 + R_1)\cdots)(1 + R_K) - 1.$$

If τ is a parameter and $\varepsilon \sim N(0, 1)$, then

$$E(e^{\tau\varepsilon}) = e^{\tau^2/2} \qquad \text{og} \qquad \text{sd}(e^{\tau\varepsilon}) = e^{\tau^2/2}\sqrt{e^{\tau^2} - 1}.$$

The Black-Scholes formula for a put option over $(0, T)$ with guarantee r_g and initial value of the stock $v_0 = 1$ is

$$\pi = (1 + r_g)e^{-rT}\Phi(a) - \Phi(a - \sigma\sqrt{T}) \qquad \text{where} \qquad a = \frac{\log(1 + r_g) - rT + \sigma^2T/2}{\sigma\sqrt{T}}.$$

Here the volatility and discount over $(0, T)$ are $\sigma\sqrt{T}$ and e^{-rT} respectively.

Problem 3 deals with sums of randomly many independent and identically distributed random variables. If $X = Z_1 + \ldots + Z_N$, $N \sim \text{Poisson}(\lambda)$ and independent of $Z_1, Z_2, \ldots$, then

$$E(\mathcal{X}) = \lambda\xi_z \qquad \text{and} \qquad \text{var}(\mathcal{X}) = \lambda(\sigma_z^2 + \xi_z^2)$$

where $\xi_z = E(Z_i)$ and $\sigma_z =$sd(Z_i). You will also need the rules of double expectation and double variance. If $\xi(\mathbf{x}) = E(Y|\mathbf{X} = \mathbf{x})$ and $\sigma^2(\mathbf{x}) = \text{var}(Y|\mathbf{X} = \mathbf{x})$ are the conditional mean and variance for Y given that a random vector $\mathbf{X} = \mathbf{x}$, then

$$E(Y) = E\{\xi(\mathbf{X})\} \qquad \text{and} \qquad \text{var}(Y) = E\{\sigma^2(\mathbf{X})\} + \text{var}\{\xi(\mathbf{X})\}.$$

Problem 1

Let $R_1, \ldots, R_K$ be returns from an investment in the stock market in each of K consecutive periods and suppose

$$R_k = e^{\xi + \sigma\varepsilon_k} - 1, \qquad k = 1, \ldots, K$$

where ξ and σ are parameters and $\varepsilon_1, \ldots, \varepsilon_K$ are independent and $N(0, 1)$.

a) What is the probability distribution of $R_{0:K}$? Argue that it is similar to that for $R_1, \ldots, R_K$, but with different parameters which you specify.

Consider a put option on $R_{0:K}$ with guaranteed return r_g. In practice such a thing may be difficult to find in the market when K is a long time horizon, but we do not take that into account.

b) If the stock is valued v_0 at time 0, what is the Black-Scholes price $\pi(v_0)$ for the put when the discount is e^{-r} in each of the K periods up to time K?

c) Argue that if you divide your initial capital v_0 on a stock investment and a fee for a put option protecting it, then the capital V at time K becomes

$$V = \{1 + \max(R_{0:K}, r_g)\}\frac{v_0}{1+\pi(1)}.$$

[**Hint:** Explain that $v_0/\{1+\pi(1)\}$ is the amount available for the stock purchase.]

It is much harder to evaluate the risk in $R_{0:K}$ when R_k earns a floating rate of interest. A possible model is

$$R_k = re^{-\tau_x^2/2+X_k} \qquad \text{where} \qquad X_k = aX_{k-1} + \tau\varepsilon_k, \quad k = 1, \dots, K$$

and $\tau_x^2 = \tau^2/(1-a^2)$. Here r, a and τ are parameters.

d). If $R_0 = r_0$ is the value of the floating rate observed in the market when the investment period starts, sketch a computer program which simulates $R_{0:K}$ now. Write the sketch in any way you like.

The table below records under three different investment strategies the mean, standard deviation and lower 5% and upper 5% percentiles of the capital V at time $K = 20$ after investing at the beginning $v_0 = 1$ (unit: 1 million NOK). The money was either placed in the stock market without derivatives, or in the same market protected by puts or in a bank account earning a floating rate of interest. Parameters and conditions were $r = 0.03$, $\xi = 0.03$, $\sigma = 0.25$, $a = 0.6$ and $\tau = 0.25$.

	$E(V)$	sd(V)	5% (lower)	5% (upper)
Stock market (no option)	3.40	5.38	0.29	11.47
Stock market (option with $r_g = 0.5$)	2.85	4.01	1.15	8.80
Stock market (option with $r_g = 2.0$)	2.36	2.55	1.55	5.93
Floating rate of interest	2.82	0.31	2.44	3.40

e) Explain how you estimate mean, standard deviation and percentiles of V from the simulations of $R_{0:K}$.

f) Write a **brief** report (just a few lines) which summarizes potential and risk when these investment strategies are used for pension saving.

Problem 2

Suppose a pension starts to run at age l_r, lasting to the end of life, with an amount s paid out at the beginning of each period. The probability of an individual of age l living at least k periods longer is ${}_kp_l$, and the discount per period is d.

a) Argue that the present value PV of the pension for an indvidual who is today at age l_0 is PV=$a_{l_0}s$ and write down a mathematical expression for a_{l_0} when you treat the cases $l_0 < l_r$ and $l_0 \geq l_r$ separately.

b) If an individual at retirement uses his savings V at that time to purchase a pension for the rest of his life, how much will he receive each period?

Suppose the individual in b) at age $l_0 < l_r$ inherits from his parents an amount v_0 which he decides to use to finance his pension by investing all of it in the stock market, earning the returns in Problem 1a).

c). Write down mathematical expressions for the mean and standard deviation of the pension he obtains.

It may be more prudent to place the money in a bank than buying shares only and another possibility is to protect the stock investment through a put option.

d) Use the table at the end of Problem 1 to compute the mean and standard deviation of the pension under the three investment strategies there and a add a few remarks to the discussion in Problem 1f) when $v_0 = 1$ and $a_{l_r} = 11.9$[1]

Many pension schemes were traditionally financed by fixed contributions ζ to support a **given** pension.

e) What does it mean that ζ is determined by equivalence? Write down a mathematical expression for it when the contributions are made at the start of each period up to one period before retirement.

The three values of ζ below apply when $s = 1$ and the contributions start at $l_0 = 47$ with other conditions as in[1] except for the discount which was varied between $d = 0.96$, $d = 0.975$ and $d = 0.99$. The order of the corresponding ζ has been changed.

$d =??$	$d =??$	$d =??$
$\zeta = 0.356$	$\zeta = 0.484$	$\zeta = 0.262$

f) Identify the values of ζ with those of the discount and justify your selection though an intutive or mathematical argument.

Problem 3

Let $\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$ be the total pay-out in a property insurance portfolio under standard assumptions where $\mathcal{N} \sim \text{Poisson}(\lambda)$ independent of the individual losses Z_1, Z_2 which are identically and independently distributed with distribution function

$$F(z) = e^{-(z/\beta)^{-\alpha}}, \qquad z > 0.$$

Here α and β are positive parameters.

[1] This value of a_{l_r} arises when $l_r = 67$, $d = 0.975$ and the life table follows the Gomperz-Makeham model

$${}_kp_l = \exp\left(-\theta_0 k - \frac{\theta_1}{\theta_2}(e^{\theta_2 k} - 1)e^{\theta_2 l}\right), \qquad k = 0, 1, \ldots,$$

with $\theta_0 = 0.009$, $\theta_1 = 0.000046$ and $\theta_2 = 0.0908$.

a) The model for the losses (named after the French mathematician Frechet) is not a standard choice, and you won't find any sampler for it in ordinary software. Program one yourself using the inversion method.

b) Write a program (a sketch) simulating $\mathcal{X}$ and explain how you determine the reserve from it.

Suppose the portfolio is reinsured through a contract where the reinsurer responsibility per event is

$$Z^{\text{re}} = \begin{matrix} 0 & \text{if } Z \leq a \\ \theta(Z-a) & \text{if } Z > a \end{matrix}$$

where a and θ are fixed by the contract and $0 \leq \theta \leq 1$.

c) Modify the program in b) so that you can determine cedent net reserve with the reinsurance reimbursement taken into account.

Cedent net responsibility are summarized in the table below when $\lambda = 30$, $\alpha = 2$ og $\beta = 1$ ($m = 100000$ simulations used), without reinsurance (first row) and with reinsurance (second row).

	mean	1%	5%	25%	50%	75%	90%	95%	99%
No reinsurance	53.1	25.6	31.7	41.7	50.8	61.9	67.6	83.0	112.7
Reinsurance[1]	47.3	24.9	30.7	39.1	46.0	53.7	62.0	67.6	83.1

[1] $a = 2.5$, $\theta = 0.5$

d) What does the cedent net reserve become with and without reinsurance?

e) Explain how you from the table can compute the pure premium of the reinurance and find the real reinsurance premium when the loading $\gamma = 1$.

Suppose the Poisson parameter $\lambda = J\mu$ where μ is the average number of claims per policy holder.

f) If μ is fixed, offer a simple matematical argument which shows that $\text{sd}(\mathcal{X})/E(\mathcal{X}) \to 0$ when $J \to \infty$.

g) Derive an expression for $\text{sd}(\mathcal{X})/E(\mathcal{X})$ when μ is random and argue that the limit of $\text{sd}(\mathcal{X})/E(\mathcal{X})$ as $J \to \infty$ is no longer 0. [**Hint:** Use the rules of double expectation and double variance.]