UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK3505/4505 — Problems and methods in Actuarial science

Day of examination: Thursday December 8th 2016

Examination hours: 14.30 – 18.30

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Introduction

Problem 1 is on general insurance, Problem 2 on life insurance and Problem 3 on financial risk. All of them must be answered and all sub-problems count equally. Write program sketches either in pseudo code or as R commands. The mathematical definitions and expressions below (that you need not prove or justify) may help you solving some of the sub-problems.

Rules of double expectation and variance:

$$\mathrm{E}(Y) = \mathrm{E}(\mathrm{E}(Y|\boldsymbol{X}))$$
$$\mathrm{Var}(Y) = \mathrm{Var}(\mathrm{E}(Y|\boldsymbol{X})) + \mathrm{E}(\mathrm{Var}(Y|\boldsymbol{X})).$$

Black-Scholes formula for a put option over $(0, T)$ with guarantee r_g and initial value of the stock $v_0 = 1$:

$$\pi(1) = (1 + r_g)e^{-rT}\Phi(a) - \Phi(a - \sigma\sqrt{T})$$

where $a = \frac{\log(1+r_g) - rT + \sigma^2 T/2}{\sigma\sqrt{T}}$. Here the volatility and the discount over $(0, T)$ are $\sigma\sqrt{T}$ and e^{-rT}, respectively.

Problem 1 General insurance

Let $\mathcal{X} = Z_1 + \ldots + Z_{\mathcal{N}}$ be the total pay-out in a general insurance portfolio under standard assumptions, where $\mathcal{N} \sim Poisson(\lambda)$ is independent of the individual losses Z_i, which are independent, identically distributed with distribution function

$$F(z) = 1 - \frac{1}{\left(1 + \left(\frac{z}{\beta}\right)^{\theta}\right)^{\alpha}}, z > 0.$$

Here α, θ and β are positive parameters.

(Continued on page 2.)

a

The model for the losses is called the Burr distribution. Derive a sampler for this distribution using the inversion method.

b

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to determine $E(\mathcal{X})$ and $sd(\mathcal{X})$, as well as the reserve from these simulations.

The table below shows the expectation, standard deviation and a few percentiles of the distribution of $\mathcal{X}$ when $\lambda = 30$, $\alpha = 1$, $\theta = 3$ and $\beta = 2$. The number of simulations was 100,000:

$E(\mathcal{X})$	$sd(\mathcal{X})$	1%	5%	25%	50%	75%	95%	99%
72.6	16.9	38.8	47.2	60.8	71.3	82.9	101.9	118.2

c

Derive expressions for $E(\mathcal{X})$ and $sd(\mathcal{X})$ when $\mathrm{E}(Z_i) = \xi(\alpha, \theta, \beta)$ and $\mathrm{Var}(Z_i) = \sigma^2(\alpha, \theta, \beta)$, which are equal to 2.42 and 3.82, respectively, for the given parameters (**Hint**: use the rules of double expectation and variance). Compare the exact values you get for $E(\mathcal{X})$ and $sd(\mathcal{X})$ with the ones from the table. Are there any signs of errors in the simulation program used for computing the values in the table?

d

What are the 95% and 99% reserves of the portfolio?

Suppose the portfolio is reinsured through a contract where the reinsurer responsibility per event is

$$Z^{re} = \begin{cases} 0, & Z \leq a \\ Z - a, & Z > a \end{cases},$$

where $a > 0$ is fixed by the contract.

e

Modify the program in **b** so that you can determine cedent net reserve with the reinsurance reimbursement taken into account.

The cedent net responsibility $\mathcal{X}^{ce}$ is summarized in the table below for $a = 2.5$ and other parameters as before.

$E(\mathcal{X}^{ce})$	$sd(\mathcal{X}^{ce})$	1%	5%	25%	50%	75%	95%	99%
56.4	10.9	32.7	39.2	48.8	56.0	63.6	74.9	83.4

(Continued on page 3.)

f

What do the 95% and 99% cedent net reserves become with reinsurance? Comment on the difference from **d**.

Assume now that the reinsurance contract now specifies a reinsurer responsibility per event of

$$Z^{re} = bZ,$$

where $0 \leq b \leq 0$ is fixed by the contract.

g

Show that $\mathcal{X}^{re} = b\mathcal{X}$ (and correspondingly $\mathcal{X}^{ce} = (1-b)\mathcal{X}$). Use that to derive the pure **reinsurance** premium and the 95% and 99% **cedent** net reserves from the table before **c** when $b = 0.223$.

The cedent net responsibility $\mathcal{X}^{ce}$ with this other reinsurance contract is summarized in the table below.

$E(\mathcal{X}^{ce})$	$sd(\mathcal{X}^{ce})$	1%	5%	25%	50%	75%	95%	99%
56.4	13.2	30.1	36.7	47.2	55.4	64.4	79.2	91.8

h

Comment on the differences between the cedent net reserves with the two different reinsurance contracts. Which contract would you prefer?

Problem 2 Life insurance

Suppose a pension starts to run at age l_r, lasting until the individual dies, with an amount s paid out at the beginning of each period. The probability of an individual of age l living at least k periods longer is ${}_kp_l$, and the discount per period is d.

a

Write down an expression for the expected present value π_{l_0} of such a pension for an individual at age l_0 when $l_0 < l_r$ and when $l_0 \geq l_r$.

The three values of π_{l_0} below apply when $s = 1$, the age of retirement is $l_r = 67$,

$$\log({}_kp_l) = -\theta_0 k - \frac{\theta_1}{\theta_2}\left(e^{\theta_2 k} - 1\right)e^{\theta_2 l},$$

with $\theta_0 = 0.009$, $\theta_1 = 0.000046$ and $\theta_2 = 0.0908$, the discount is 0.975 and l_0 is varied between $37, 47, 57$. The order of the corresponding π_{l_0} has been changed.

l_0	??	??	??
π_{l_0}	4.98	7.40	3.46

(Continued on page 4.)

b

Identify the values of l_0 with those of π_{l_0} and justify your argument.

c

Assume that the pension is financed by fixed contributions ζ at the start of each year from age $l_0 < l_r$ up to age $l_r - 1$. What is the expected present value of all these payments?

d

What does it mean that ζ is determined by equivalence? Write down a mathematical expression for it when the contributions are made at the start of each period up to one period before retirement.

Problem 3 Financial risk

The random walk on logarithmic scale is a common model for equity. This means that the stock price S_k at time k is given by

$$S_k = e^{Y_k}$$
$$Y_k = Y_{k-1} + X_k, k = 1, 2, \ldots,$$

where $X_k = \xi + \sigma\epsilon_k$, $S_0 = v_0$ and the ϵ_ks are independent and $\sim N(0, 1)$.

a

What is the K-step return $R_{0:K}$ and what is its probability distribution? (**Hint:** show first that $R_k = e^{X_k} - 1$).

Assume that the investment is protected by a put option on $R_{0:K}$ with guaranteed return r_g.

b

Sketch a program for simulating the pay-off $X = \max(r_g - R_{0:K}, 0)$, and explain how you could use that to compute the risk-neutral price $\pi(v_0)$ when the discount is e^{-r} in each of the K periods (remember that in the risk-neutral model, $\xi = r - \frac{1}{2}\sigma^2$)? Can you think of an alternative method for computing $\pi(v_0)$ in this case?

Suppose the original investment v_0 instead is spread on J stocks, and that we are only considering one time period. The assumed model for the returns is now

$$R_j = \xi_j + \sigma_j\epsilon_j, \ \epsilon_j \sim N(0, 1),$$

and the portfolio return is $\mathcal{R} = w_1R_1 + \ldots + w_JR_J$, with $w_1 + \ldots + w_J = 1$.

(Continued on page 5.)

c

Assume first, somewhat unrealistically, that $\xi_j = \xi$, $\sigma_j = \sigma$, $w_j = \frac{1}{J}$, $j = 1, \ldots, J$, and that $\mathrm{Cor}(\epsilon_i, \epsilon_j) = 0$, $i \neq j$. Show that

$$\frac{\mathrm{sd}(\mathcal{R})}{\mathrm{E}(\mathcal{R})} \underset{J\to\infty}{\longrightarrow} 0.$$

d

Assume now that all stock returns depend on the same market return R_M via $\xi_j = r + \beta(R_M - r)$, where r is constant and $R_M = \xi_M + \sigma_M \epsilon_M$, with $\epsilon_M \sim N(0,1)$ independent of $\epsilon_1, \ldots, \epsilon_J$. Assume that the remaining conditions are as in **c** and derive the limiting value of $\frac{\mathrm{sd}(\mathcal{R})}{\mathrm{E}(\mathcal{R})}$ as $J \to \infty$ (**Hint**: use the rules of double expectation and variance). Comment on the difference from **c**.

END