Exam in STK3505/4505: Answers

Problem 1

a

The moment estimates are given by:

$$\frac{\hat{\beta}}{\hat{\alpha}-1} = \bar{z} \quad \text{and} \quad \bar{z}\sqrt{\frac{\hat{\alpha}}{\hat{\alpha}-2}} = s,$$

such that

$$\hat{\alpha} = \frac{2}{1-\bar{z}^2/s^2} = 4.01 \quad \text{and} \quad \hat{\beta} = \bar{z}\,(\hat{\alpha}-1) = 8.04.$$

The variance is only defined when $\alpha > 2$. Therefore, moment estimation cannot be used for the Pareto distribution when $\alpha \leq 2$, and one may also run into numerical problems when α is > 2, but close to 2, i.e. when the Pareto distribution is very heavy-tailed.

b

Simulation of Z using the inversion method requires an expression for the inverse cdf $F^{-1}(u)$:

$$F(z) = 1 - \frac{1}{\left(1+\frac{z}{\beta}\right)^{\alpha}} = u$$
$$z = \beta\left((1-u)^{-1/\alpha} - 1\right) = F^{-1}(u).$$

Simulation algorithm:

1: Input: α, β
2: Draw $U^* \sim U(0,1)$
3: Return $Z^* = \beta\left((1-U^*)^{-1/\alpha} - 1\right)$ % or $\beta\left((U^*)^{-1/\alpha} - 1\right)$

c

Simulation algorithm for $\mathcal{X}$:

1: Input: $\lambda, \alpha, \beta, m$
2: **for** i=1,...,m **do**
3: Draw $\mathcal{N}^* \sim Poisson(\lambda)$
4: $\mathcal{X}_i^* \leftarrow 0$
5: **for** j=1,...,$\mathcal{N}^*$ **do**

Draw $Z^* \sim Pareto(\alpha, \beta)$
$\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$
end for
end for
Return $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$.

Estimate of the mean: $\bar{\mathcal{X}}^* = \frac{1}{m}\sum_{i=1}^{m} \mathcal{X}_i^*$.
Estimate of the standard deviation: $s_{\mathcal{X}}^* = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(\mathcal{X}_i^* - \bar{\mathcal{X}}^*\right)^2}$.
The $100 \cdot \epsilon\%$ reserve q_ϵ is given by

$$\mathrm{P}(\mathcal{X} \leq q_\epsilon) = \epsilon,$$

and estimated by $q_\epsilon^* = \mathcal{X}_{(m\epsilon)}^*$, where $\mathcal{X}_{(1)}^* \leq \ldots \leq \mathcal{X}_{(m)}^*$.

d
$\mathrm{E}(\mathcal{X}) = \lambda\frac{\beta}{\alpha-1} = 80.1$.
$\mathrm{sd}(\mathcal{X}) = \frac{\beta}{\alpha-1}\sqrt{\lambda\left(\frac{\alpha}{\alpha-2}+1\right)} = 25.3$.

e
The 95% and 99% reserves are 125.5 and 153.0, respectively.

f
Simulations of cedent net portfolio payoffs $\mathcal{X}_1^{ce,*}, \ldots, \mathcal{X}_m^{ce,*}$ are obtained by applying:
$\mathcal{X}_i^{ce,*} = \mathcal{X}_i^* - \min\left(\max\left(\mathcal{X}_i^* - a, 0\right), b\right)$
to the simulations from **c**. The corresponding $100 \cdot \epsilon\%$ cedent net reserve is estimated by $q_\epsilon^{ce,*} = \mathcal{X}_{(m\epsilon)}^{ce,*}$, where $\mathcal{X}_{(1)}^{ce,*} \leq \ldots \leq \mathcal{X}_{(m)}^{ce,*}$.

g
The cedent net 95% and 99% reserves are 90.3 and 118.4, respectively. Obviously, the reserves become lower when part of the responsibility is transferred on a reinsurer. Since there is an upper bound on this responsibility which is close to the 95% reserve without reinsurance, the reduction in the 99% reserve is smaller than that of the 95% reserve. The reinsurance pure premium is $\pi^{re,pu} = \mathrm{E}(\mathcal{X}^{re}) = \mathrm{E}(\mathcal{X}) - \mathrm{E}(\mathcal{X}^{ce}) = 5.2$ and the actual premium is $\pi^{re} = (1+\gamma)\pi^{re,pu} = 9.36$.

Problem 2

a

$$\pi_{l_0} = \begin{cases} s\sum_{k=l_r-l_0}^{\infty} d^k \;{}_k p_{l_0}, & l_0 < l_r \\ s\sum_{k=0}^{\infty} d^k \;{}_k p_{l_0}, & l_0 \geq l_r \end{cases},$$

where ${}_k p_{l_0}$ takes the risk of death into account, whereas d^k is the discount factor.

b

The shorter the time between the start age l_0 and the age of retirement l_r, the less the pension is discounted. Further, the probability of surviving til and past the age of retirement increases with increasing age. Hence, the present value increases with l_0. This explains the upwards shape of the curves in the plot. Further, a higher value of the discount rate r leads to a heavier discount of the pension payments in the future, i.e. smaller discount factors d^k, and hence a smaller present value of the pension. This means that the solid, dashed, dotted and dash-dotted curves correspond to $r = 0.05$, 0.04, 0.03 and 0.02, respectively.

c

Present value of payments: $\zeta\sum_{k=0}^{l_r-l_0-1} d^k \;{}_k p_{l_0}$, where as in **a** ${}_k p_{l_0}$ takes the risk of death into account, whereas d^k is the discount factor.

d

Equivalence means that the expected present value of the payments ζ should be equal to the expected present value of the pension π_{l_0}, so that:

$$\zeta = s\frac{\sum_{k=l_r-l_0}^{\infty} d^k \;{}_k p_{l_0}}{\sum_{k=0}^{l_r-l_0-1} d^k \;{}_k p_{l_0}}.$$

Problem 3

a

$R_k = \frac{S_k}{S_{k-1}} - 1 = \frac{e^{Y_k}}{e^{Y_{k-1}}} - 1 = e^{Y_k - Y_{k-1}} - 1 = e^{X_k} - 1.$

K-step return: $R_{0:K} = (1+R_1)\cdot \ldots \cdot (1+R_K) - 1.$

$R_{0:K} = (1+R_1)\cdot \ldots \cdot (1+R_K) - 1 = e^{X_1}\cdot \ldots \cdot e^{X_K} - 1 = e^{\sum_{j=1}^{K} X_j} - 1 = e^{K\xi + \sigma\sum_{j=1}^{K}\varepsilon_j}.$

Let $\eta = \frac{1}{\sqrt{K}} \sum_{j=1}^{K} \varepsilon_j$. Since $\varepsilon_j \overset{iid}{\sim} N(0,1)$, η is also normally distributed with

$$\mathrm{E}(\eta) = \frac{1}{\sqrt{K}} \sum_{j=1}^{K} \mathrm{E}(\varepsilon_j) = 0$$
$$\mathrm{Var}(\eta) = \frac{1}{K} \sum_{j=1}^{K} \mathrm{Var}(\varepsilon_j) = 1.$$

Thus, $R_{0:K} = e^{K\xi + \sigma\sqrt{K}\eta} - 1$, with $\eta \sim N(0,1)$.

b
Simulation algorithm for X:

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Input: r, σ, v_0, r_g, K
{ R* ← 0
{ P* ← 1
for k=1,...,K do
    Draw ε* ~ N(0,1)
    R* ← e^{ξ+σε*} − 1
    P* ← P*(1 + R*)
end for
Return R*_{0:K} ← P* − 1.
```

With m simulations $R^*_{0:K,1}, \ldots, R^*_{0:K,m}$ from the above algorithm, the pdf of $R_{0:K}$ can be estimated by

$$f^*(R) = \frac{1}{mh} \sum_{i=1}^{m} K\left(\frac{R - R^*_{0:K,i}}{h}\right),$$

where $K(\cdot)$ is a kernel function with mean 0 that integrates to 1, for instance a Gaussian kernel. This can be seen as a smoothing of a histogram, where the degree of smoothness is controlled by the bandwidth parameter h. Small values of h give a low bias, but a high variance and a un-smooth estimate. Higher values of h give a higher bias, but also a lower variance and a smoother estimate. The choice of bandwidth should therefore be a trade-off between bias and variance.

c As the ε_ks follow a normal distribution, so does $X_k|x_0$. This means that $e^{X_k}|x_0$ follows a log-normal distribution with parameters $\left(a^k x_0, \sqrt{\frac{1-a^{2k}}{1-a^2}}\tau\right)$, and its mean and standard deviation are as given in the formulas on the first page of the exam. We have:

$$\begin{aligned}
\mathrm{E}(r_k|r_0) &= re^{-\tau^2/(2(1-a^2))}\mathrm{E}(e^{X_k}|x_0) \\
&= re^{-\tau^2/(2(1-a^2))}e^{a^k x_0 + \frac{1-a^{2k}}{2(1-a^2)}\tau^2} \\
&= re^{a^k x_0 - \tau^2 a^{2k}/(2(1-a^2))}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{sd}(r_k|r_0) &= re^{-\tau^2/(2(1-a^2))}\mathrm{sd}(e^{X_k}|x_0) \\
&= re^{-\tau^2/(2(1-a^2))}e^{a^k x_0 + \frac{1-a^{2k}}{2(1-a^2)}\tau^2}\sqrt{e^{\frac{1-a^{2k}}{1-a^2}\tau^2} - 1} \\
&= \mathrm{E}(r_k|r_0)\sqrt{e^{\tau^2(1-a^{2k})/(1-a^2)} - 1}.
\end{aligned}$$

d The stock investment gives a high expected return and has a very high upside, with a 95% quantile of 5.50. The bank account with a floating interest rate obviously gives a much lower expected return and upside. On the other hand, the risk is also much lower, which is seen both on the standard deviation, that is much lower, and the 5% quantile, that is much higher than for the stock return. The choice of investment must therefore depend on the risk appetite of the investor.