UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Answers

Day of examination: ??. ??. ????

This problem set consists of 6 pages.

Appendices: None

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 General insurance

1a

The moment estimates are given by

$$\frac{\hat{\beta}}{\hat{\alpha}-1} = \bar{z}$$

and

$$\bar{z}\sqrt{\frac{\hat{\alpha}}{\hat{\alpha}-2}} = s,$$

such that

$$\hat{\alpha} = \frac{2}{1-\frac{\bar{z}^2}{s^2}} = 4.02$$

and

$$\hat{\beta} = \bar{z}(\hat{\alpha}-1) = 12.71$$

The variance is only defined when $\alpha > 2$. Therefore, moment estimation cannot be used for the Pareto distribution when $\alpha \leq 2$, and one may also run into numerical problems when $\alpha > 2$, but close to 2, i.e. when the Pareto distribution is very heavy-tailed.

1b

Simulation of Z using the inversion method requires an expression for the inverse cdf $F^{-1}(u)$:

$$F(z) = 1 - \frac{1}{\left(a+\frac{z}{\beta}\right)^{\alpha}} = u$$

(Continued on page 2.)

$$z = \beta\left((1-u)^{-\frac{1}{\alpha}} - 1\right) = F^{-1}(u).$$

Simulation algorithm:

1. Input: α, β
2. Draw $U^* \sim U(0,1)$
3. Return $Z^* = \beta\left((1-U^*)^{-\frac{1}{\alpha}} - 1\right)$ or $Z^* = \beta\left((U^*)^{-\frac{1}{\alpha}} - 1\right)$

1c

Simulation algorithm for $\mathcal{X}$:

1. Input $\alpha, \beta, \lambda, m$
2. for $i = 1, \ldots, m$ do
3. Draw $\mathcal{N}^* \sim Poisson(\lambda)$
4. $\mathcal{X}_i^* \leftarrow 0$
5. for $j = 1, \ldots, \mathcal{N}^*$ do:
6. Draw $X^* \sim Pareto(\alpha, \beta)$
7. $\mathcal{X}_i^* \leftarrow \mathcal{X}_i^* + Z^*$
8. end for
9. end for
10. Return $\mathcal{X}_1^*, \ldots, \mathcal{X}_m^*$.

Estimate of the mean: $\overline{\mathcal{X}^*} = \frac{1}{m}\sum_{i=1}^m \mathcal{X}_i^*$.

Estimate of the standard deviation: $S^*_{\mathcal{X}^*} = \sqrt{\frac{1}{m-1}\sum_{i=1}^m (\mathcal{X}_i^* - \overline{\mathcal{X}^*})^2}$

The $100 * \epsilon\%$ reserve r_ϵ is given by $P(\mathcal{X} \leq q_\epsilon = \epsilon$ and estimated by $q_\epsilon^* = \mathcal{X}^*_{(m\epsilon)}$, where $\mathcal{X}^*_{(1)} \leq \cdots \leq \mathcal{X}^*_{(m)}$.

1d

Using double expectation and double variance to show that.

$$E(\mathcal{X}^*) = E(E(\mathcal{X}^*|\mathcal{N})) = E(\mathcal{N}\frac{\beta}{\alpha-1}) = \lambda\frac{\beta}{\alpha-1} = 84.2$$

$$var(\mathcal{X}^*) = var(E(\mathcal{X}^*|\mathcal{N})) + E(var(\mathcal{X}^*|\mathcal{N})) = var(\mathcal{N}\frac{\beta}{\alpha-1}) + E\left(\mathcal{N}(\frac{\beta}{\alpha-1})^2\frac{\alpha}{\alpha-2}\right) =$$

(Continued on page 3.)

$$= \lambda \left(\frac{\beta}{\alpha - 1}\right)^2 + \lambda \left(\frac{\beta}{\alpha - 1}\right)^2 \frac{\alpha}{\alpha - 2}$$

$$sd(\mathcal{X}^*) = \frac{\beta}{\alpha - 1}\sqrt{\lambda\left(\frac{\alpha}{\alpha - 2} + 1\right)} = 32.56$$

1e

The 95% and 99% reserves are 142, 71 and 180.01.

1f

Simulations of cedent net portfolio payoffs $\mathcal{X}_1^{ce,*}, \ldots, \mathcal{X}_m^{ce,*}$ are obtained by applying:

$$\mathcal{X}_i^{ce,*} = \mathcal{X}_i^* - min(max(\mathcal{X}_i^* - a, 0), b)$$

to the simulations from **c**. The corresponding $100 * \epsilon\%$ cedent net reserve is estimated by $q_\epsilon^{ce,*} = \mathcal{X}_{(m\epsilon)}^{ce,*}$, where $\mathcal{X}_{(1)}^{ce,*} \leq \cdots \leq \mathcal{X}_{(m)}^{ce,*}$.

1g

The cedent net 95% and 99% reserves are 110.0 and 131.8, respectively. Obviously, the reserves become lower when part of the responsibility is transferred on a reinsurer.

The reinsureance pure premium is $\pi^{re,pu} = E(\mathcal{X}^{re}) = \mathcal{X} - \mathcal{X}^{ce} = 3.9$ and the actual premium is $\pi^{re} = (1+\gamma) * \pi^{re,pu} = 6.63$.

Problem 2 Life insurance

2a

$$\pi^{l_0} = \begin{cases} s\sum_{k=l_r-l_0}^{\infty} d^k {}_kp_{l_0} & , l_0 < l_r, \\ s\sum_{k=0}^{\infty} d^k {}_kp_{l_0} & , l_0 \geq l_r, \end{cases}$$

where ${}_kp_{l_0}$ takes the risk of death into account and d^k is the discount factor.

2b

The longer the time between the start age l_0 and the age of retirement l_r, the more the pension is discounted, and the smaller the present value. Thus, π_{l_0} is an increasing function of l_0, so the order is: 57, 37, 47

2c

Present value of payments: $\zeta \sum_{k=0}^{l_r-l_0-1} d^k {}_kp_{l_0}$.

(Continued on page 4.)

2d

Equivalence means that the expected present value of the payments ζ should be equal to the expected present value of the pension π_{L_0} , so that:

$$\zeta = s\frac{\sum_{k=l_r-l_0}^{\infty} d^k {}_k p_{l_0}}{\sum_{k=0}^{l_r-l_0-1} d^k {}_k p_{l_0}}$$

Problem 3 Financial risk

3a

The premium for put option in terms of single asset is

$$\pi(v_0) = e^{-rT} E_Q(max(r_g - R, 0))v_0,$$

where $R = e^{\xi_q T + \sigma\sqrt{T}\epsilon} - 1$ for $\epsilon \sim N(0,1)$. There is positive payoff if $R < r_g$ or equivalently if $\epsilon < a$ where

$$a = \frac{log(1+r_g) - \xi_q T}{\sigma\sqrt{T}}$$

and the option premium becomes

$$\pi(v_0) = e^{-rT}\left(\int_{-\infty}^{a}(1 + r_g - e^{\xi_q T + \sigma\sqrt{T}x})\phi(x)dx\right).$$

Splitting the integrand gives

$$\pi(v_0) = e^{-rT}\left((1+r_g)\int_{-\infty}^{a}\phi(x)dx - e^{\xi_q T}\int_{-\infty}^{a} e^{\sigma\sqrt{T}x}\phi(x)dx\right)$$

where the second integral on the right is

$$\int_{-\infty}^{a} e^{\sigma\sqrt{T}x}(2\pi)^{-\frac{1}{2}}e^{-\frac{x^2}{2}}dx = e^{\sigma^2\frac{T}{2}}\int_{-\infty}^{a} 2\pi)^{-\frac{1}{2}}e^{-frac(x-\sigma\sqrt{T})^2 2}dx.$$

If $\Phi(x) = \int_{-\infty}^{x}\phi(y)dy$ is the Gaussian integral then

$$\pi(v_0) = e^{-rT}((1+r_g)\Phi(a) - e^{\xi_q T + \sigma^2\frac{T}{2}}\Phi(a - \sigma\sqrt{T}))v_0$$

and inserting $\xi_q = r - \frac{\sigma^2}{2}$ into the expression and into a yields the Black-Scholes formula.

(Continued on page 5.)

3b

1. Input: parameters r_g, r, σ, T, m etc
2. for $i = 1, \ldots, m$ do:
3. Draw $N(0,1)$
4. $R_i^* \leftarrow e^{rT-\sigma^2\frac{T}{2}+\sigma\sqrt{T}\epsilon} - 1$
5. $X_i^* \leftarrow max(r_g - R_i^*, 0)$
6. end for
7. return $\pi^* = \frac{e^{-rT}}{m}\sum_{i=1}^m X_i^*$

3c

$X_P = max(r_g - R, 0)\nu_0$ and $X_C = max(R - r_g, 0)\nu_0$.
If $r_g > R$ then $X_P = (r_g - R)\nu_0$ and $X_C = 0$ so $X_C - X_P = (R - r_g)\nu_0$.
If $r_g < R$ then $X_P = 0$ and $X_C = (R - r_g)\nu_0$ so $X_C - X_P = (R - r_g)\nu_0$.
$\pi_{P/C} = e^{-rT}E_Q(X_{P/C})$ so

$$\pi_C(\nu_0) - \pi_P(\nu_0) = e^{-rT}E_Q(X_C) - e^{-rT}E_Q(X_P) = e^{-rT}(E_Q(R) - r_g)\nu_0.$$

Using similar method as before (on the exam you need to write the derivations) we get that $E_Q(R) = -1 + e^{rT}$ so

$$\pi_C(\nu_0) - \pi_P(\nu_0) = e^{-rT}(-1 + e^{rT} - r_g)$$

and finally

$$\pi_C(\nu_0) = pi_P(\nu_0) = +(1 - e^{-rT(1+r_g)})\nu_0$$

3d

We know that $r_c > r_g$

$$X_P(r_g) - X_C(r_c) = max(r_g - R, 0)\nu_0 - max(R - r_c, 0)\nu_0 =$$

$$\begin{cases} if R \le r_g: & (r_g - R)\nu_0 - 0 = (r_g - R)\nu_0, \\ if r_g < R \le r_c: & 0 - 0 = 0 \\ if R > r_c: & 0 - (R - r_c)\nu_0 \end{cases}$$

and so we get the definition of a cliquet option.

The cliquet option lowers the cost by allowing the option seller to keep the top of the return. Any return above a ceiling r_c is kept by the seller. The quarantee is still r_g but the instrument is cheaper.

(Continued on page 6.)

3e

X can be priced as $e^{-rT}E_Q(X) = e^{-rT}E_Q(X_P(r_g) - X_C(r_c)) = e^{-rT}E_Q(X_O(r_g)) - e^{-rT}E_Q(X_C(r_c)) = \pi_P(r_g) - \pi_C(r_c)$. We know from the previous exercises that $\pi_C = \pi_P + (1 - e^{-rT}(1 + r_g))v_0$ so $\pi = \pi_P(r_g) - \pi_C(r_c) - (1 - e^{-rT}(1 + r_c))v_0$

END