UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK3505/4505 — Problems and methods in Actuarial science

Day of examination: 27th November 2019

Examination hours: 09:00 – 13:00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 is on general insurance, Problem 2 on life insurance and Problem 3 on financial risk. All of them must be answered and all sub-problems count equally. Write program sketches either in pseudo code or as R commands.

Problem 1 General insurance

Let $\mathcal{X} = Z_1 + \cdots + Z_{\mathcal{N}}$ be the pay-out in a general insurance portfolio under standard assumption, where $\mathcal{N} \sim Poisson(\lambda)$ is independent of the individual losses Z_i, which are independent, identically $Pareto(\alpha, \beta)$ distributed.

If $Z \sim Pareto(\alpha, \beta)$, then

$$E(Z) = \frac{\beta}{\alpha - 1} \text{ and } sd(Z) = E(Z)\sqrt{\frac{\alpha}{\alpha - 2}}$$

for $\alpha > 1$ and $\alpha > 2$, respectively, and

$$F(z) = 1 - \frac{1}{\left(1 + \frac{z}{\beta}\right)^{\alpha}} \text{ for } z > 0.$$

Assume that for a given set of historical losses $z_1, \ldots, z_n$, the sample mean and standard deviation are 4.21 and 5.94, respectively.

1a

Find the moment estimates of α and β. What difficulties may occur when using this method of estimation for the Pareto distribution?

1b

Derive a sampler for the Pareto distribution using the inversion method.

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1c

Sketch a program for generating m samples of $\mathcal{X}$. Explain how to determine $E(\mathcal{X})$ and $Var(\mathcal{X})$, as well as the reserve from these simulations. Define what the reserve is.

The table below shows the expectation, standard deviation and a few percentiles of the distribution of $\mathcal{X}$ when $\lambda = 20$. The number of simulations was 100000:

$E(\mathcal{X})$	$sd(\mathcal{X})$	1%	5%	25%	50%	75%	95%	99%
84.13	32.73	27.36	39.55	61.10	79.92	102.08	142.71	180.01

1d

Derive expressions for $E(\mathcal{X})$ and $sd(\mathcal{X})$ using the rules of double expectation and variance. Compare the exact values of $E(\mathcal{X})$ and $sd(\mathcal{X})$ with the ones from the table. Are there any signs of errors in the simulation program?

1e

What are the 95% and 99% reserves of the portfolio?

Suppose the portfolio is reinsured through a contract where the reinsurer responsibility is

$$\mathcal{X}^{re} = \begin{cases} 0 & \text{if } \mathcal{X} \leq a, \\ \mathcal{X} - a & \text{if } a < \mathcal{X} \leq a + b, \\ b & \text{if } \mathcal{X} > a + b, \end{cases}$$

where $a, b > 0$ are fixed by the contract.

1f

How is the net responsibility of the cedent simulated?

The cedent net responsibility $\mathcal{X}^{ce}$ is summarized in the table below for $a = 110$, $b = 50$ and other parameters as before.

$E(\mathcal{X})$	$sd(\mathcal{X})$	1%	5%	25%	50%	75%	95%	99%
80.23	25.37	28.05	39.51	61.47	79.99	102.10	110.00	131.80

(Continued on page 3.)

1g

What are the 95% and 99% **cedent** net reserves? Comment on the differences from **e**. What is the **reinsurance** pure premium and what is the actual reinsurance when the loading is $\gamma = 0.7$.

Problem 2 Life insurance

Suppose a pension starts to run at age l_r, lasting until the individual dies, with an amount s paid out at the beginning of each period. The probability of an individual of age l living at least k periods longer is ${}_kp_l$, and the discount per period is d.

2a

Write down an expression for the expected present value π_{l_0} of such a pension for an individual at age l_0 when $l_0 < l_r$ and when $l_0 \geq l_r$.

The three values of π_{l_0} below apply when $s = 1$, the age of retirement is $l_r = 67$,

$$\log({}_kp_l) = -\theta_0 k - \frac{\theta_1}{\theta_2}\left(e^{\theta_2 k} - 1\right) e^{\theta_2 l},$$

with $\theta_0 = 0.009$, $\theta_1 = 0.000046$ and $\theta_2 = 0.0908$, the discount is 0.975 and l_0 is varied between $37, 47, 57$. The order of the corresponding π_{l_0}has been changed.

l_0	??	??	??
π_{l_0}	7.40	3.46	4.98

2b

Identify the values of l_0 with those of π_{l_0} and justify your argument.

2c

Assume that the pension is financed by fixed contributions ζ at the start of each year from age $l_0 < l_r$ up to age $l_r - 1$. What is the expected present value of all these payments?

2d

What does it mean that ζ is determined by equivalence? Write down a mathematical expression for it when the contributions are made at the start of each period up to one period before retirement.

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Problem 3 Financial risk

Consider a put option with pay-off $X_P = \max(r_g - R, 0)$, and a call option with pay-off $X_C = \max(R - r_g, 0)$, where $R = e^{\xi_q T + \sigma\sqrt{T}\epsilon} - 1$, $\epsilon \sim N(0,1)$, r_g is the minimum return, r is a risk-free interest rate, T is the time period and ξ_q, σ are parameters. The put option has premium $\pi(\nu_0) = e^{-rT} E_Q\{(X_P)\}\nu_0.$, where Q is the risk-neutral measure, and ν_0 is the initial capital.

3a

Derive the Black-Scholes formula for the put option premium.

3b

Sketch a program for simulating premium of the put option.

3c

Argue that $X_C - X_P = (R - r_g)\nu_0$ and use this to deduce that option premia $\pi_P(\nu_0)$ and $\pi_C(\nu_0)$ are related through

$$\pi_C(\nu_0) = \pi_P(\nu_0) + \{1 - e^{-rT}(1 + r_g)\}\nu_0.$$

Consider now a cliquet option with pay-off

$$\begin{cases} r_g - R & \text{if } R \leq r, \\ 0 & \text{if } r_g < R \leq r_c, \\ -(R - r_c) & \text{if } R > r_c, \end{cases}$$

where r_c is the ceiling.

3d

Options can be treated also as a function of the guaranteed return rate r_g so that $X_P := X_P(r_g)$. Argue that the payoff for cliquet options can be written as $X = X_P(r_g) - X_C(r_c)$. What is the benefit of cliquet options?

3e

Deduce from previous exercises that the premium for a cliquet option becomes $\pi = \pi_P(r_g) - \pi_P(r_c) - \{1 - e^{-rT}(1 + r_c)\}\nu_0$.

END