Problems and Methods in Actuarial Science

Autumn 2023, UiO

Formalities

- **Classes:** Monday and Wednesdays 8:25-10:00 (5 min break)
- **Textbook**: Erik Bølviken "Computation and Modelling in Insurance and Finance", Cambridge University Press, 2014
- **Exam:** December 6th at 15:00 (h). **Location:** Silurveien 2 Sal 3C
- **Mandatory assignment:** October 19th, 14:30 (will come back to it)
- **Website:** **https://www.uio.no/studier/emner/matnat/math/STK3505/h23/index.html**
- Presentations, solutions of exercises and other info will be published on the website regularly

Formalities, continued

- The course will cover the material from the textbook, but will be presented in slightly different order and manner. Information on the material covered in each week on the website.
- The lectures will extend/supplement the textbook with some basic notions about insurance and info about real life problems. Most of it will NOT be required on the exam.
- Material required during the exam will be marked in the final presentation with the red rectangular in the corner.

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Actuarial science

What it is?

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> Actuarial science is the discipline that applies **mathematical and statistical methods** to assess risk in insurance, pension, finance, investment and other industries and professions. More generally, actuaries apply rigorous mathematics to model matters of **uncertainty**.

https://en.wikipedia.org/wiki/Actuarial_science

Actuarial science includes a number of interrelated subjects, including **mathematics**, **probability theory**, **statistics**, **finance**, **economics**, **financial accounting** and **computer science**.

https://en.wikipedia.org/wiki/Actuarial_science

“ Historically, actuarial science used **deterministic models** in the construction of tables and premiums. The science has gone through revolutionary changes since the 1980s due to the proliferation of high speed computers and the union of **stochastic actuarial models** with **modern financial theory**.

https://en.wikipedia.org/wiki/Actuarial_science

What is differentiating *actuarial science* from *mathematical finance* is the focus on **risk sharing and diversification** in order to mitigate the adverse effect of events of uncertain occurrence, severity and timing.

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Structure of the course

1. Introduction
2. Finance, Part 1:
   - Time value of money
   - Monte Carlo,
   - Distributions and basic models for assets
3. Non-Life insurance:
   - Compound Poisson model and Monte Carlo
   - Premium calculation
   - Reserving
   - Reinsurance
   - How it works in practice
4. Life insurance
   - Life-tables, discounting under mortality
   - Reserves and premium calculation
   - Markov chains in life-insurance
   - Monte Carlo simulation
   - Modern insurance products
5. Finance, Part 2:
   - Combining Investments and Insurance results - > ruin problem
   - Finding optimal investment portfolio
   - Options and Black-Scholes formula
   - Price of guaranteed returns in life insurance products via options

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Insurance

What is it and how does it work?

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Book: Chapter 1, sections 1.1–1.2

Basic idea behind insurance –

sharing the risk across many individuals in order to be able to cover unlikely but substantial losses

Works because of the Law of Large Numbers

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Basic mathematic model

- $X_1, X_2, \dots, X_n$ - losses from policy 1,2,…,n (assumed to be i.i.d.)
- $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} X_i = E(X_1) =: \pi^{pu}$ **pure premium**
- Insurance company in this model does not lose (or profit) almost surely if it charges pure premium and has infinite amounts of customers
- In practice:
  - no portfolio is infinite
  - losses can occur at the same time (e.g. NatCats, Covid-19),
  - different policies with different risk (but this is less of a problem by proper risk differentiation)
  - Insurance companies want to have some profit
- Therefore: premium paid has a loading γ such that $\pi = \pi^{pu}(1+\gamma)$

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Insurance

Non-life insurance/ Property & Casualty (P&C)

Usually **1-year** cover against unlikely risks with adverse financial effect

- Car insurance (3rd party liability, casco)
- Property (building, contents)
- Travel insurance
- Workers Compensation (yrkesskade)
- Health-related products
- Specialistic covers (liability, pet, marine, agriculture, business interruption)
- (Collective) death insurance

Life insurance

Usually **longer term contracts** which protect against events where the uncertainty is both **whether** and **when** the trigger event occurs

- Death insurance/Term-insurance
- Pure endowment insurance
- Endowment insurance
- Pension – "insurance against living for too long time" ;-)
- Disability insurance
- Nowadays also: saving and investment without any particular "insurance" element

Role of an actuary

Pricing actuary

- Building models which predict **risk level given risk factors (**e.g. age of a driver)
- Determining premium level for different groups (**tariffs**) based on those models
- Risk selection (what to insure, what not to)
- Follow up of the exposure to detect changes and update models

Reserving actuary

- Building models to set up **reserves**
- Reserves cover **expected future payments** for claims which have not happened, or which have happened but the amount to be paid is uncertain
- Compare the estimates from pricing against experience

Capital actuary

- Building models to determine how much capital to hold to cover **unexpected losses**, i.e. future payments which exceeds reserves
- Advising about the risk appetite (i.e. how much risk to take given available capital) and how to use capital efficiently

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Accounting in the actuarial science

Why?

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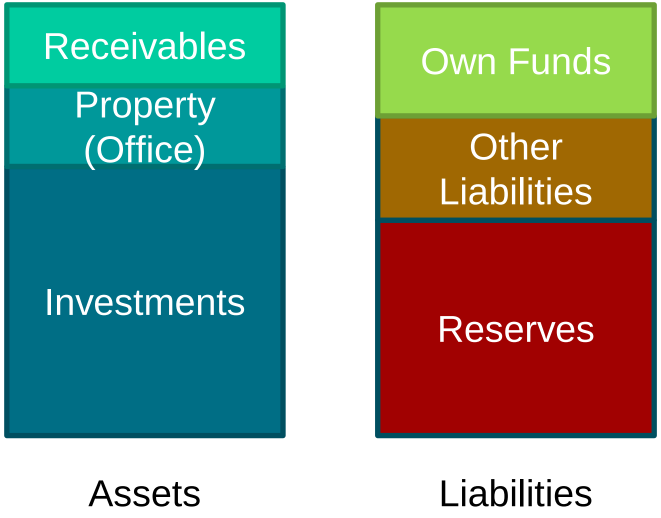
Balance sheet of an insurance company



Assets: what a company "owns"
- Investments, property, cash
- "Receivables", i.e. what others owe to an insurance company, e.g. not paid premiums

Liabilities: what a company owes to others
- **Reserves** (AKA "**technical provisions**")
- Other liabilities (e.g. loans or debt)

Equity or **Own Funds:** net wealth
- Simplistically: assets minus liabilities
- Can be seen as what a company owes to its owners

Change in equity = profit = investment return + profit on UW +...

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Why an actuary needs to know a bit of accounting?

Pricing actuary in Non-Life:

- not really that important ☺

Reserving and capital actuaries, pricing actuaries in Life:

- reserves are one of the main elements on the liability side and determine level of equity and profit
- Different standards and rules
  - IFRS (International Financial Reporting Standard)
  - Local GAAP (Generally Accepted Accounting Principles)
  - Solvency or tax purposes
- For Life-insurance: interaction between profits, reserves and benefits.

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Finance and the actuarial science

What's the connection here?

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Why finance?

- Reserves can be seen as **cashflows** (i.e. a sequence of future payments/money transfers happening at different times). Therefore natural to treat them in financial terms and e.g. to discount them.
- Investments – appropriate to the nature of the reserves and contracts
  - More about that during the first exercise classes
- Interaction between reserves and investment returns, especially for life insurance
  - Cashflows may depend on the result from investments. Then the amount of the cashflow can be seen as a financial derivative/contingent claim

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Life-insurance products as saving / investment

Term-insurance

- Pays an agreed amount (benefit) if insured dies in an agreed period
- Insurance against dying
- Uncertain if benefit is paid and when

Pure endowment

- Pays a benefit if insured is alive at the end of an agreed period
- Insurance against surviving
- Uncertain if benefit is paid, no uncertainty about timing

Endowment

- Combination of term and pure endowment
- Pays a benefit regardless if insured dies or survives it
- No uncertainty regarding the occurrence of benefit, but only about timing

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Endowment insurance as a form of saving/ investment

Assume single premium $\pi = 1000$ paid at the beginning of the contract and benefit B=1300, paid out if an insured dies before next 15 years or after 15 years if an insured survives.

If the insured survives, then she will yield an annually compounded return

$$\sqrt[15]{B/\pi} - 1 = 1{,}76\%$$

If the insured dies after 5 years, then her heirs will yield an annually compounded return equal to

$$\sqrt[5]{B/\pi} - 1 = 5{,}39\%$$

Risk sharing here is between those who survive and get possibly lower returns, and those who die and whose heirs get higher returns.

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Time value of money

I.e. why $1000 today is not the same as $1000 in a year

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Book, Chapter 1, sections 1.3, 1.4 (without 1.4.3)

Concept

- An amount of money today is worth more* now than the same amount will be worth in the future
- This is due to the fact that we can invest the money today and earn potential profit on this investment
- To compare to amounts of money at two different time points we need to take that profit into account

* We assume here that interest rates are non-negative ;-)

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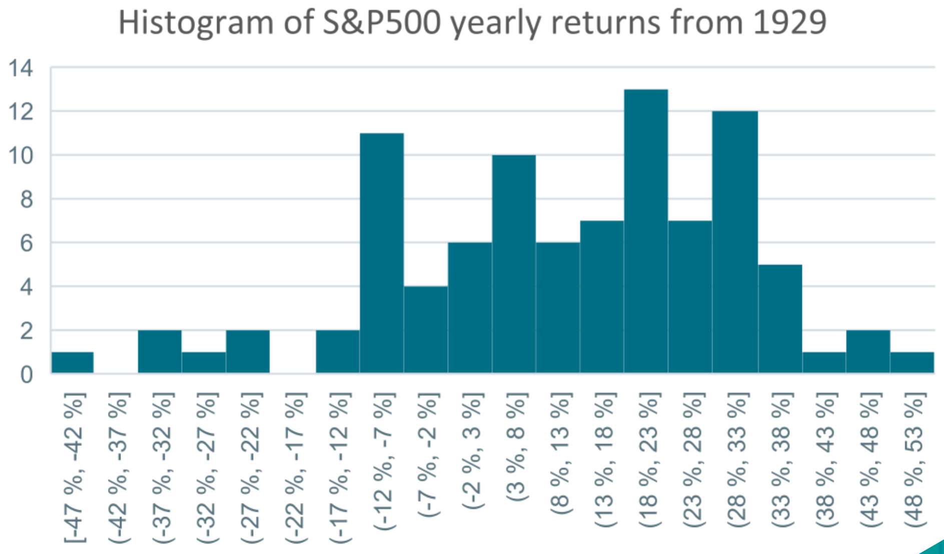
Risk-free interest rate vs. risky assets

- Most of the assets are risky, i.e. we don't really know their future value nor return
- If we invest $1000 e.g. in stocks today equally plausible that in a year we have $900 or $1100. Difficult to use risky assets to assess time-value of money.
- We need to use an asset with a "risk-free" return, e.g. sovereign bonds or "bank account".



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Principal, interest and compounding

- **Principal** or capital is an amount of money which the interest is calculated upon. E.g. I open a saving account in a bank and put 1000 kr on it. The principal then is 1000 kr.
- **Interest** is an amount of money which you earn on principal by holding it on that saving account. E.g. That saving account offers 12% yearly interest, so after a year I will earn 12%*1000 = 120 kr.
- **Compounding** interest (or capitalization) of is time after which the earned interest is added to the principal. E.g. if 12% is earned with yearly compounding, after a year the principal will grow to 1120 kr.

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On interest rate and return – annual compounding

- Assume we put \$1000 on a bank account with interest rate 12% and yearly capitalization.
- After a year we will have therefore \$1120. If the interest rate is unchanged and we keep all the money and the proceeds on the account, after another year we will get \$1254,4, i.e. 12% of \$1120.

Annual compounded interest rate:

$$V_K = V_0(1+r)^K$$

V_t - Value of the bank account at time t (in years), t=0,1,2,3,…

r – interest rate (with annual capitalization of interest)

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Present value of a cashflow – discounting with the annually compounded interest

- We know that the amount X at time 0 is worth $X(1+r)^K$ after K years
- How much is the amount Y which we can get in K years, worth now?

$$\frac{Y}{(1+r)^K}$$

- This is known as a **present value (PV)** of a future cashflow or **discounting** of a future cashflow

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On interest rate and return – monthly compounding

- Assume we put $1000 on a bank account with yearly interest rate 12% and monthly capitalization.
- After a month we will have therefore $1010, i.e. 12% /12. If the interest rate is unchanged and we keep all the money and the proceeds on the account, after another month we will get $1020,1, i.e. 1% of $1010.
- After two years (with interest rate unchanged) we would get $1269,735

Montly compounded interest rate:

$$V_K = V_0\left(1+\frac{r}{12}\right)^{K*12}$$

V_t - Value of the bank account at time t (in years), $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots$

r – annual interest rate (with monthly capitalization of interest)

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On interest rate and return – daily and continuous compounding

Similarly, in case of daily capitalization the value of the bank account can be calculated as

$$V_K = V_0(1 + r * p)^{K/p}, where$$

$p = 1/365$ is the fraction of the year after which the interest is capitalized

What would happen if interest is capitalized instantaneously? That would mean $p \to 0$

$$\lim_{p \to 0} V_0(1 + r * p)^{K/p} = V_0 e^{rK} = V_K$$

Therefore $1000 after 2 years with 12% continuously compounded interest is:

$1000e^{2*12\%}$=$ 1271,25

Note also that we can measure the value at any non-negative time K.

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Annual/continuous compounding vs. arbitrage

Arbitrage = profit without risk

Assume annually compounded and continuously compounded interest rates are both12%. Then I could borrow $1000 with annual compounding for two years and put it on the bank account with continuous compounding for two years.

After two years I need to return $1254,4, but I have $1271,25 on my bank account. I.e. without taking any risk I gained $16,85, i.e. managed to find arbitrage opportunity.

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Annual/continuous compounding vs. arbitrage

Markets with arbitrage shouldn't exist, and therefore annually compounded interest rate must be higher than continuously compounded interest rate.

Assume that a annually compounded interest rate r_a is 12%. What is the continuously compounded interest rate r_c which leads to arbitrage-free market?

We know that that after K years with annually compounded interest we have $V_0(1+r_a)^K$ whereas with continuous compounding we get $V_0 e^{r_c K}$. To avoid arbitrage both amounts must equal.
Easy to check that: $r_c = \ln(1+r_a)$=11,33%.

Note that we can also discount with continuous compounding, but we need to use the proper interest rate!!!!

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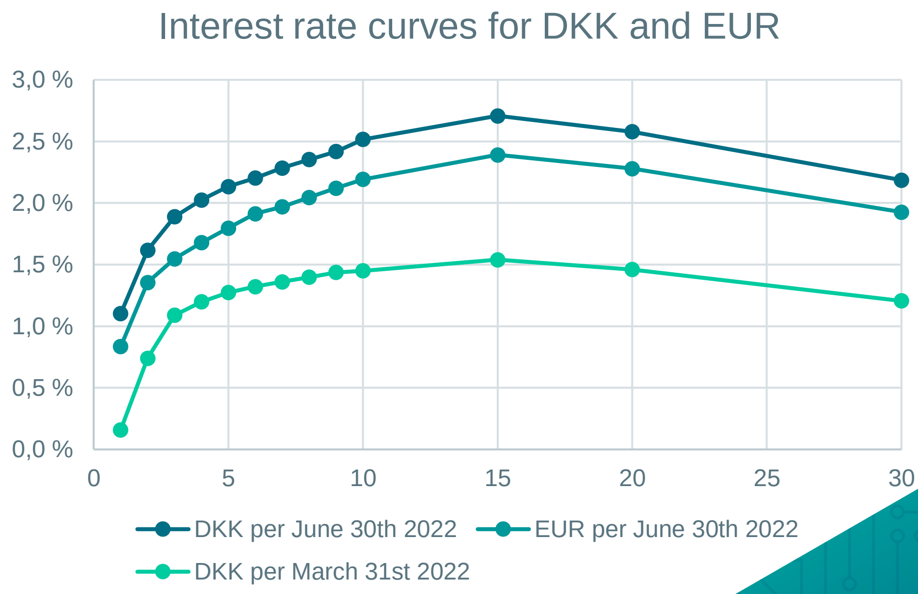
Is r the same regardless timing of cashflow? – zero coupon bonds

- Rate on the bank account can change (“refinancing risk”) so no reason to assume that
- **Zero-coupon bond** – an instrument which pays you at maturity K an amount 1 (and nothing earlier or later). We assume no risk, i.e. we get 1 with probability 1
- $P_0(0:K)$ - price (at time 0) of zero-coupon bond with maturity K, we have it from the market
- $P_0(0:K) = 1 * \left(1 + \bar{r}_0(0:K)\right)^{-K}$
- So: $\bar{r}_0(0:K) = \left(P_0(0:K)\right)^{-1/K}$-1
- $\bar{r}_0(0:K)$ not necessarily equal $\bar{r}_0(0:L), K \neq L$
- **Interest rate curve**: $t \mapsto \bar{r}_0(0:t)$
- Discounting with market interest rates leads to **fair valuation** or **market value** of a cashflow



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Bonds and duration

- **Bond** (in general) is an instrument paying a fixed stream of payments $B_1, B_2, \dots, B_K$ at some predefined times $t_1, t_2, \dots, t_K$.
- Bond = "portfolio of zero-coupon bonds"
- **Value of the bond**:

$$V_0 = \sum_{i=1}^{K} B_i * P_0(0: t_i)$$

- **Yield** of the bond: the solution (in y) of

$$V_0 = \sum_{i=1}^{K} \frac{B_i}{(1+y)^{t_i}}$$

If interest rate curve flat yield y = interest rate r.

This is what we mostly assume in this course!!!!

Duration of a bond (or sequence of cashflows): average time when payments occur, weighted with the discounted size of the payment

$D = \sum_{i=1}^{K} t_i * q_i$, where $q_i = \frac{B_i(1+y)^{-t_i}}{\sum_{i=1}^{K} B_i(1+y)^{-t_i}} = \frac{B_i(1+y)^{-t_i}}{V_0}$.

Note that the price changes if interest rate/yield changes. What is the sensitivity of the price to the yield?
Easy to prove that:

$$\frac{d}{dy} V_0 = \frac{-D}{1+y} V_0.$$

Therefore: $\Delta V_0 \approx \frac{-D}{1+y} V_0 * \Delta y$.

Interpretation: Take a bond with duration 5 years and price $V_0 = 100$. Assume r=y=3%. If interest rate increases with 1 **percentage point** (i.e. to 4%), the price of the bond will decrease by approximately $\frac{D}{1+y}$ **percent** (i.e. 5/(1,03) %) . New price is 100*(1-5/1,03*0,01))=95,15.

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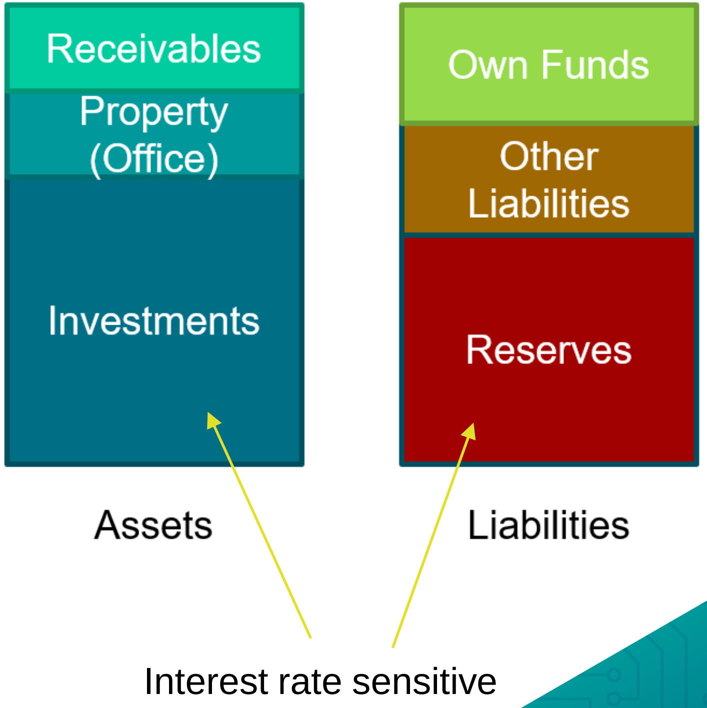
Insurance reserves as cashflows

- We mentioned that reserves for the future benefits or claims can be seen as a sequence of cashflows.
- If we assume that this sequence of cashflows is deterministic, we can treat it just like a bond and discount, compute duration, interest rate sensitivity.
- Using duration, we can therefore measure how much our assets and liabilities change by change of interest rate and therefore how much our Own Funds (=Assets – Liabilities) vary



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Risky assets, portfolios and returns

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Risky assets

- So far we mostly considered risk-free "fixed" interest rate and zero-coupon bonds
- Most of assets carry risk regarding future value:
  - Risky bonds (e.g. corporate bonds)
  - Stocks
  - Property
  - Commodities
- Even the future value of a zero-coupon bond is not constant/deterministic before the maturity. Only at maturity we know for sure the value of a zero-coupon bond (i.e. 1).
- Let V_0 be a value of an asset at time 0. Let V_1 be a value of the same asset at time 1.
- The return R of this asset is defined as relative change in the value, i.e. $R = \frac{V_1 - V_0}{V_0}$.
- At time 0 we don't know V_1 so it is a random variable, and therefore R is a random variable too.
- At time 1, we know the realisation of $V_1(\omega)$. Therefore we know therefore also realisation of $R(\omega)$.

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Portfolios and multi-period returns

We can have a **portfolio** (i.e. basket) of different assets. Let V_0^j be a value invested in an asset j at time 0.

Then $V_0 = \sum_{j=1}^{J} V_0^j$ and $w_j = \frac{V_0^j}{V_0}$ is the weight of asset j in the portfolio (at time 0)

Let $V_1 = \sum_{j=1}^{J} V_1^j$ be the value at time 1.

Easy to prove that $V_1 = \sum_{j=1}^{J}(1 + R^j)w_j V_0$, where R^j is a return for asset j.

In other words $\mathrm{R} = \sum_{j=1}^{J} R^j w_j$

Similarly we can consider multi-period returns.
Let V_k value of the investment in an asset at time k.
Let R_k a return which that assets earns between time (k-1) and k.

Then $V_K = V_0(1 + R_1)(1 + R_2) \ldots (1 + R_K)$.

Overall return between 0 and K (*K-step return* in the book) can be expressed as:

$$R_{0:K} = (1 + R_1)(1 + R_2) \ldots (1 + R_K) - 1$$

If the step is a year, we can be interested in average annual return $\bar{R}_{0:K}$. Then

$$\bar{R}_{0:K} = [(1 + R_1)(1 + R_2) \ldots (1 + R_K)]^{\frac{1}{K}} - 1 \text{ and}$$

$$(1 + \bar{R}_{0:K})^K = (1 + R_{0:K}).$$

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Lets simplify it a bit with log-returns

Let $X_k = \ln(1+R_k)$ – a **log-return** in period between k-1 and k.
Let $X_{0:K} = \ln(1+R_{0:K})$

Easy to check that $X_{0:K} = \mathrm{X}_1 + \cdots + \mathrm{X}_\mathrm{K}$
Similarly, Let $\bar{X}_{0:K}$ be the average annual log-return in period between 0 and K, defined as $\bar{X}_{0:K} = \ln(1+\bar{R}_{0:K})$.
Then

$$\bar{X}_{0:K} = \ln(1+\bar{R}_{0:K}) = \ln\left([(1+R_1)(1+R_2)\dots(1+R_K)]^{\frac{1}{K}}\right) = \frac{1}{K}[\ln(1+R_1)+\cdots+\ln(1+R_K)] =$$

$$= \frac{1}{K}[\mathrm{X}_1+\cdots+\mathrm{X}_\mathrm{K}]$$

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Monte Carlo simulation

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Book, chapter 2, sections 2.1-2.2 and 2.3.2

Monte Carlo simulation

Monte Carlo method is an algorithm based on repeated sampling/simulating a "system" in order to investigate the resulting distribution of results.

The idea:

1. We want to investigate a random vector X, however we don't know its distribution
2. We know however there is a random vector Y and a function G such that Y is easy to sample from and X=G(Y).
3. So lets simulate Y by sampling $Y_1^*, Y_2^*, \dots, Y_m^*$, calculate $X_i^* = G(Y_i^*)$ and investigate the properties of $X_1^*, X_2^*, \dots, X_m^*$

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Monte Carlo mean and standard dev.

MC mean

Estimator $\overline{X}^* = \frac{1}{m}\sum_{i=1}^{m} X_i^*$ is an unbiased estimator, i.e. $E[\overline{X}^*] = \mu$, where $\mu = E[X]$.

By LLN the estimator is also consistent (i.e. converges to the true value as m goes to infinity).

Easy to check that $sd[\overline{X}^*] = \frac{\sigma}{\sqrt{m}}$, where $\sigma = sd(X)$.

MC standard deviation

Estimator $s^* = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}(X_i^* - \overline{X}^*)^2}$ is an asymptotically unbiased estimator of true standard deviation σ.

It is also consistent and approximately $sd[s^*] \approx \frac{\sigma}{\sqrt{2m}}\sqrt{1 + \kappa/2}$, where $\kappa = excess\ kurtosis(X)$.

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About estimation error

It can be much easier to estimate correctly second order estimates (e.g. st.deviation) than first-order parameters (e.g. mean), especially if true mean is much lower than the true standard deviation

$$\frac{sd[\bar{X}^*]}{\mu} = \frac{\sigma}{\mu}\frac{1}{\sqrt{m}}$$

$$\frac{sd[s^*]}{\sigma} \approx \frac{1}{\sqrt{m}}\sqrt{\frac{1}{2} + \kappa/4}$$

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Building confidence intervals - mean

Both MC mean and standard deviation approximately normal as $m \to \infty$.

Then we know that $(1-\alpha)$-confidence interval can for MC mean be defined as

$\overline{X}^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}} < \mu < \overline{X}^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}}$, where $U_{\frac{\alpha}{2}}$ is $(1-\frac{\alpha}{2}) - quantile$ of a standard normal distribution,

This is called $(1-\alpha)$-confidence interval as

$$P\left(\overline{X}^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}} < \mu < \overline{X}^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{m}}\right) \approx 1-\alpha.$$

Note that $U_{2,5\%} \approx 1{,}96 \ldots \approx 2$, therefore formula from the book for 95%-confidence interval is the same as here.

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Similarly standard deviation

Then we know that $(1-\alpha)$-confidence interval for MC standard deviation can be defined as $s^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}}\sqrt{1+\kappa/2} < \sigma < s^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}}\sqrt{1+\kappa/2}$, where $U_{\frac{\alpha}{2}}$ is $(1-\frac{\alpha}{2}) - quantile$ of a standard normal distribution,

Similarly to mean

$$P\left(s^* - U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}}\sqrt{1+\kappa/2} < \sigma < s^* + U_{\frac{\alpha}{2}} * \frac{s^*}{\sqrt{2m}}\sqrt{1+\kappa/2}\right) \approx 1-\alpha.$$

Note that $U_{2,5\%} \approx 1{,}96 \ldots \approx 2$, therefore formula from the book for 95%-confidence interval is the same as here.

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Percentile/quantile

α −**quantile** of the distribution with CDF F is defined as an amount q_α such that

$$F(q_\alpha) = \alpha$$

Percentile is formally the same but α is expressed as percentage, i.e. 5-percentile is 5%-quantile.

The book introduces lower and upper percentiles/quantiles, but we will just focus on lower quantiles.

Quantiles measure the tail of the distribution, i.e. what is the worst loss we can expect with the given probability.

α −quantile in financial world is often called often Value-at-Risk (VaR) at a given confidence level.

Solvency Capital Requirement is based on VaR of a change of Own Funds (i.e. loss of capital) on 1-yr basis on confidence level 0,5% (i.e. 1-in-200 year loss).

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MC quantile

Assume we have MC simulations $X_1^*, X_2^*, \dots, X_m^*$. We would like to estimate α −quantile from that simulation set.

The algorithm:

1. First calculate so-called **order statistics** $X_{(1)}^*, X_{(2)}^*, \dots, X_{(m)}^*$, i.e. find such a sequence $i_1, i_2, \dots, i_m$ such that
   - $X_{(i_k)}^* = X_k^*$ and
   - $X_{(1)}^* \geq X_{(2)}^* \geq \cdots \geq X_{(m)}^*$
2. We define MC quantile q_α^* as

$$q_\alpha^* = X_{(m(1-\alpha))}^*$$

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MC properties

q_α^* is asymptotically unbiased and consistent, i.e. expected value of q_α^* converges to the true value as $m \to \infty$ as well as q_α^* converges to the true value almost surely

Error:

$$sd(q_\alpha^*) \approx \frac{\sqrt{\alpha(1-\alpha)}}{f(q_\alpha)} \frac{1}{\sqrt{m}},$$

Where f is a density function of X. This is unknown but we can have some estimates.

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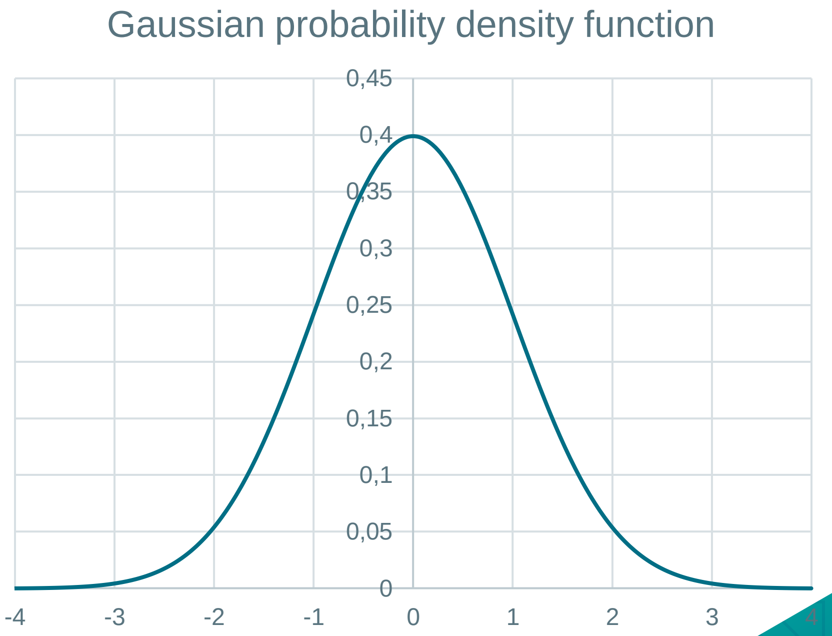
Density estimate with Gaussian kernel

Gaussian density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Idea: try to estimate the density of X from the MC simulation set by weighting appropriately Gaussian densities



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MC density estimate

$f^*(x) = \frac{1}{m}\sum_{i=1}^{m} \frac{1}{hs^*}\phi\left(\frac{x-X_i^*}{hs^*}\right)$, where h is a smoothing parameter.

$f^*(x)$ is a biased estimator with

$$E\left(f^*(x)\right) \approx f(x) + \frac{1}{2}h^2\sigma^2\frac{d^2 f(x)}{dx^2}$$

And

$$sd\left(f^*(x)\right) \approx \frac{0.5311}{\sigma}\sqrt{\frac{f(x)}{hm}}$$

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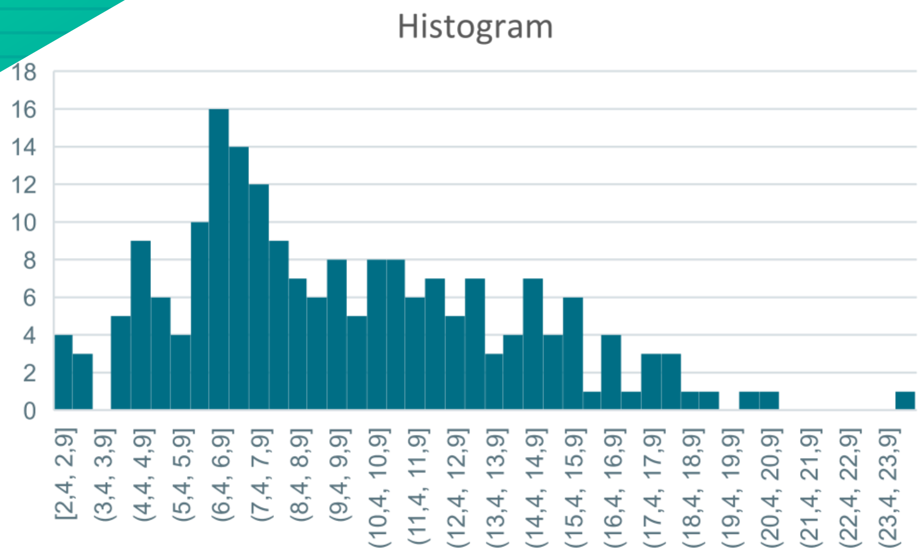
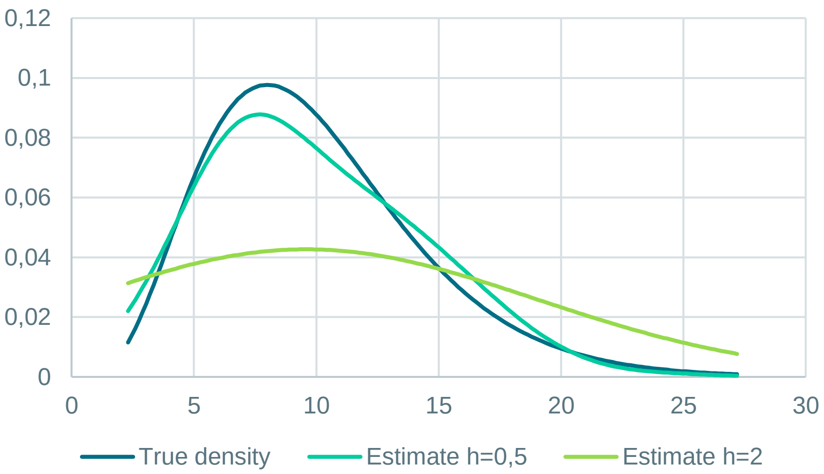
Example, S*=4,2





Classified: General Business

Sampling using the inverse method

Given a distribution F how to draw random number from that distribution?

It can be proved that the strictly increasing cumulative distribution function F has an inverse $F^{-1}(x)$ such that $\mathrm{F}\left(F^{-1}(x)\right) = x$.

First note that $F(X) \sim U$ as

$$P(F(X) \leq u) = P\left(X \leq F^{-1}(u)\right) = F\left(F^{-1}(u)\right) = \mathrm{u} \text{ if } \mathrm{u} \in (0,1)$$

Therefore $\mathrm{X} \sim F^{-1}(U)$.

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Sampling algorithm for inverse method

1. Input: The percentile function $\left(F^{-1}(x)\right)$
2. Draw $U^{*} \sim Uniform$
3. Return $X^{*} \sim F^{-1}(U^{*})$ or $X^{*} \sim F^{-1}(1 - U^{*})$

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MC and finance

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Book: Chapter 2, section 2.4

Gaussian/normal distribution

Density function for normal distribution with mean μ and standard deviation σ

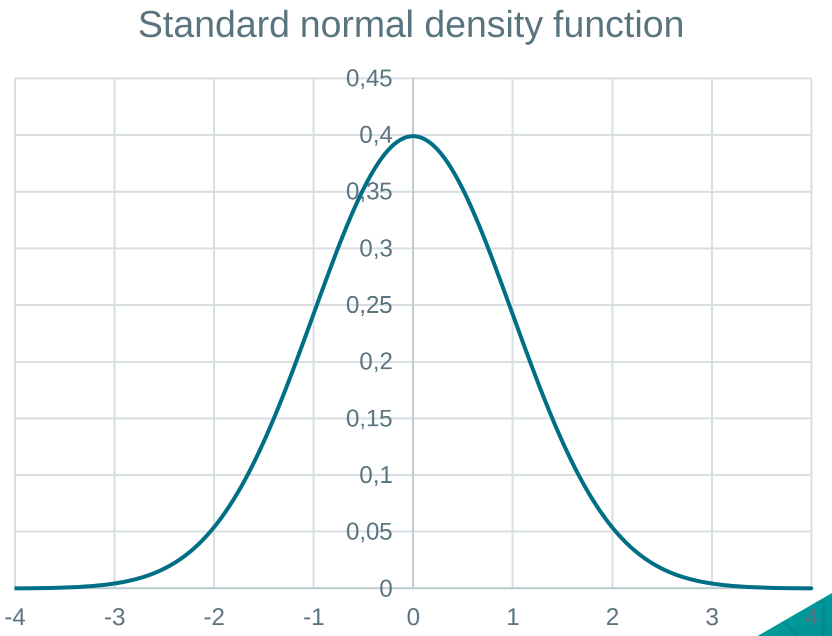
$$\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Closed form for CDF does not exist, however good algorithms to generate both CDF and its inverse.

Therefore possible to generate random variables via inverse method.

In R:

pnorm, dnorm, qnorm, rnorm



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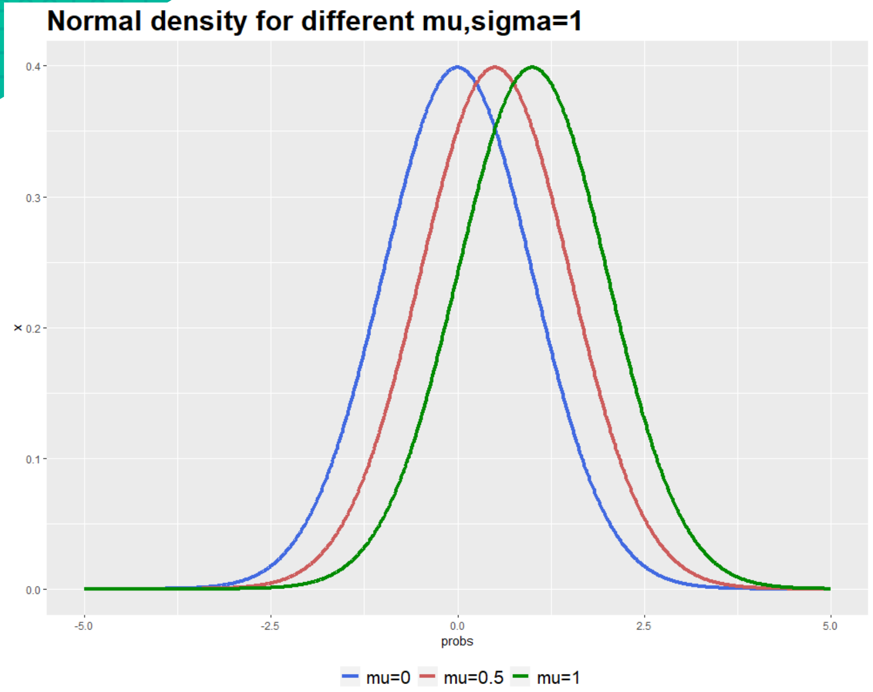
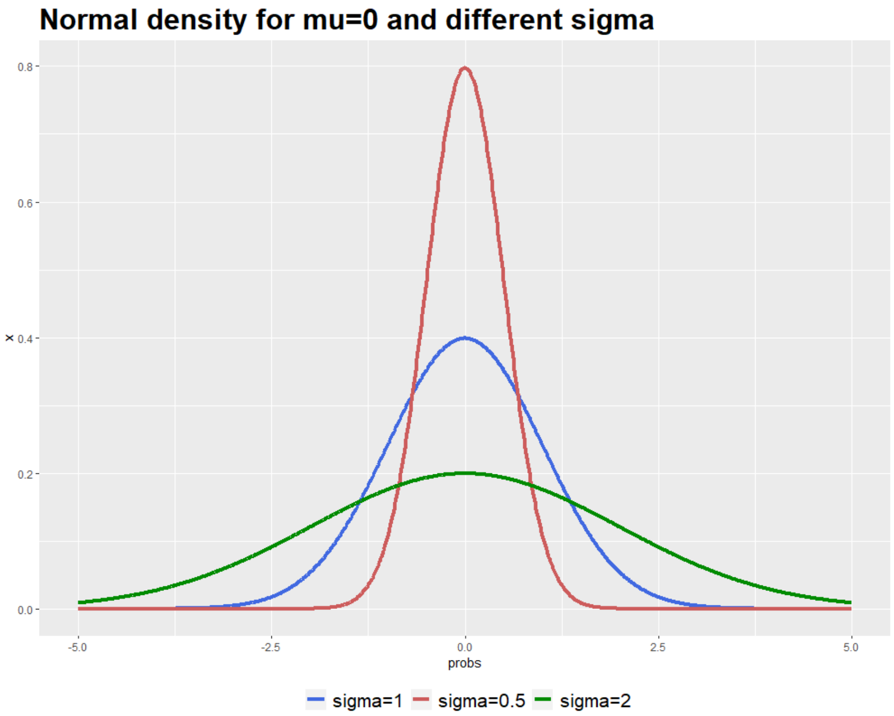
Effect of the parameter changes





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Log-normal distribution

- Very common distribution such that $X = \exp(Y)\,, Y \sim N(\mu, \sigma)$. We say then that X is log-normally distributed with paramters μ and σ.
- $E(X) = e^{\mu + \frac{\sigma^2}{2}}$
- $sd(X) = E(X)\sqrt{e^{\sigma^2} - 1}$
- Distribution has a positive skew and fat right tail
- Drawing easy via normal distribution
- Density

$$\phi(x) = \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}, \mathrm{x} > 0.$$

- In R: dlnorm, plnorm, qlnorm, rlnorm

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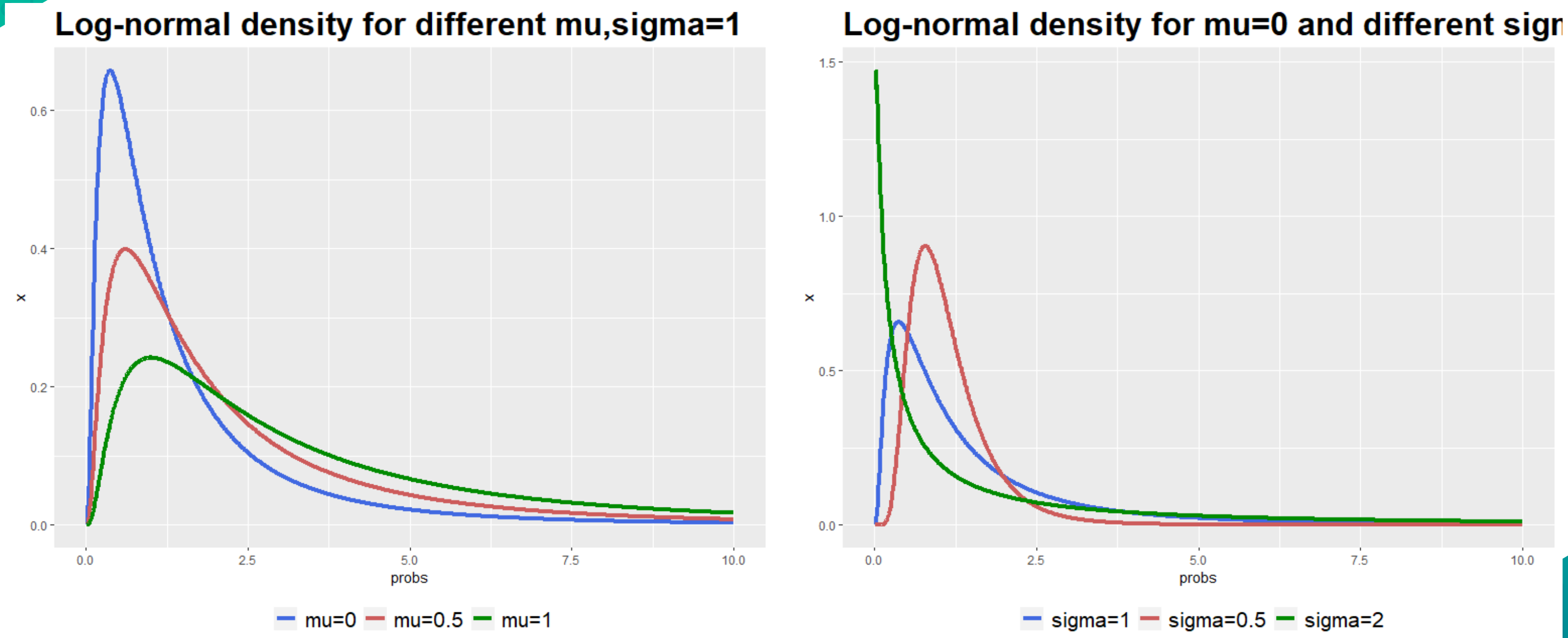
Effect of the parameter change



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Normal log-returns

Assume that R is a return on an asset and $X = \ln(1 + R)$ its log-return.
Assume that $X \sim N(\mu, \sigma)$. Then

$$R = \exp(X) - 1$$

is distributed according to (shifted) log-normal distribution with parameters μ and σ.

Basic asset models (like Black-Scholes) assume log-normal distribution of asset price

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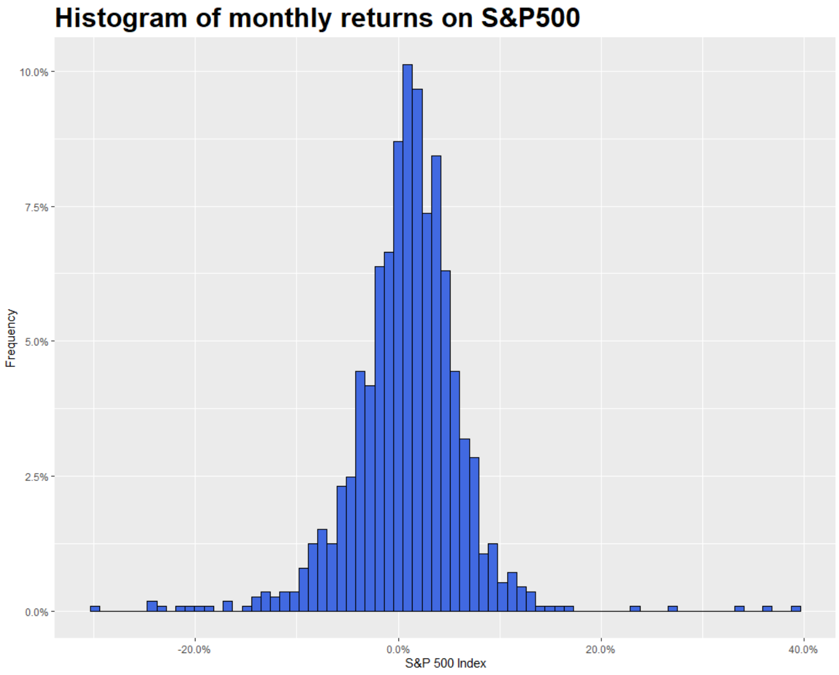
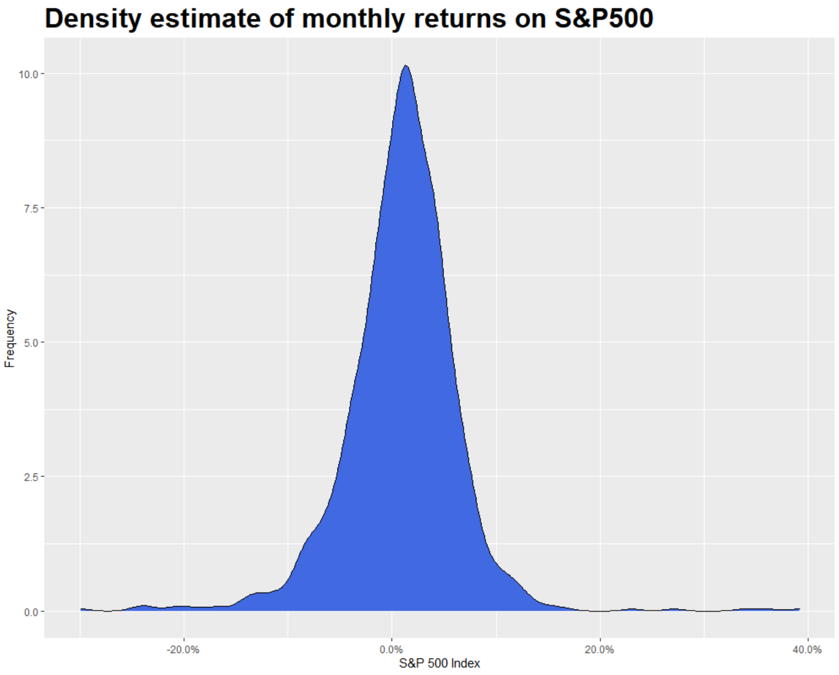
Monthly logreturns on S&P500 Index





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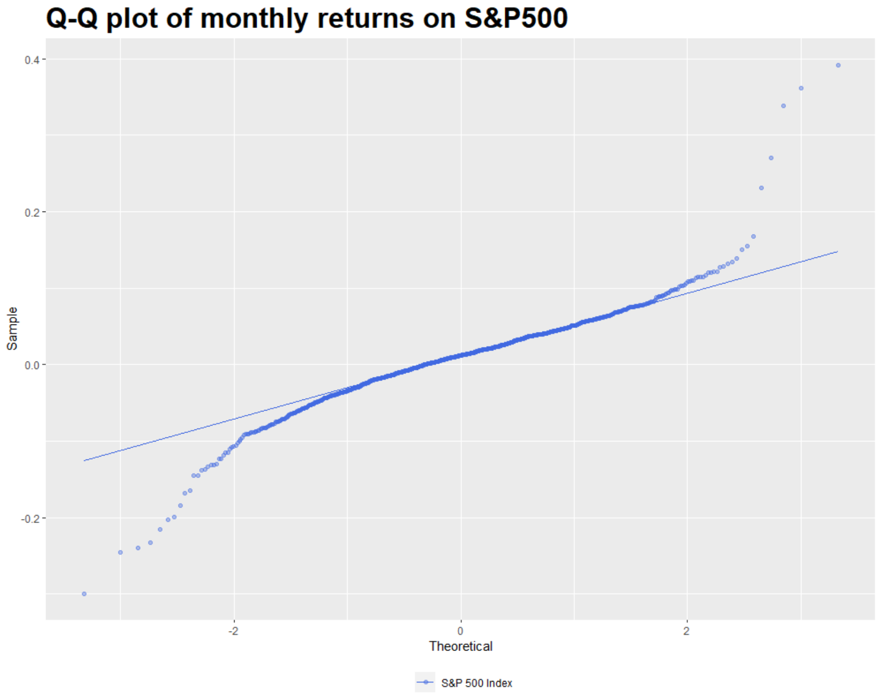
Monthly logreturns on S&P500 Index



0.05-quantile	0.25-quantile	median	mean	0.75-quantile	0.95-quantile	St.dev
-7.6%	-1,6%	1,2%	0,9%	3,8%	8.0%	5,4%

- Clear sign of non-normality
- Negative skew (i.e. distribution is assymmetric and left tail is thicker and median is higher than mean)
- Fat tails

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t-distribution

- Normality of log-returns problematic, due to tails (and asymmetry)
- Partial solution: a distribution with heavier tails like Student's t-distribution,
- In the book: t-distribution defined via Gamma-distribution.
- Let $G \sim Gamma(\alpha, \alpha)$, i.e. shape α and rate α such that mean =1
- Then if $\mathrm{Z} \sim N(0,1)$ and G are two independent random variables, then $\mathrm{X} = \frac{\mathrm{Z}}{\sqrt{G}}$ is distributed according to t-distribution with α-degrees of freedom
- $E(X) = 0, sd(X) = \sqrt{\frac{\alpha}{\alpha-2}}$. α>2 (error in the book!!!!), excess kurtosis= $\frac{6}{\alpha-4}, \alpha > 4$
- If $\alpha \rightarrow \infty$ then t-distribution converges to normal distribution
- In R: pt, dt, qt, rt from library mvtnorm

Classified: General Business

T-distribution

- More common, via χ^2-distribution.
- If $X_1, X_2, \dots, X_K \sim N(0,1)\ i.i.d.$ Then $\sum_{i=1}^{K} X_i^2$ has χ^2-distribution with K-degrees of freedom (denoted $\chi^2(K)$).
- If $\mathrm{Z} \sim N(0,1)$ and $\mathrm{S} \sim \chi^2(K)$ are two independent random variables then $\frac{\mathrm{Z}}{\sqrt{S/K}}$ has t-distribution with K-degrees of freedom
- Easy connection with the test statistics for normal distribution for mu if sigma is unknown

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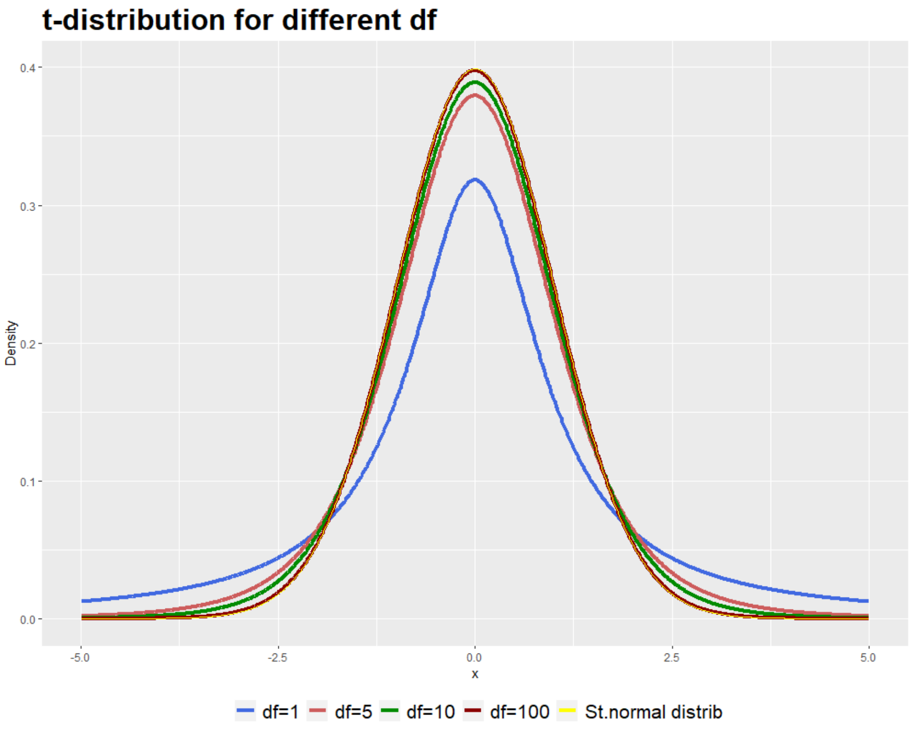
Effect of the parameter change



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Returns based on t-distribution

- Let $\mu \in \mathbb{R}, \xi_\sigma > 0, Z \sim N(0,1), G \sim Gamma(\alpha, \alpha) - independent$
- Define return R as shifted and scaled t-distribution, i.e.:

$$R = \mu + \xi_\sigma \frac{Z}{\sqrt{G}} = \mu + \sigma Z$$

- $\sigma := \frac{\xi_\sigma}{\sqrt{G}}$ can be seen as **stochastic volatility** (volatility=variability/risk)

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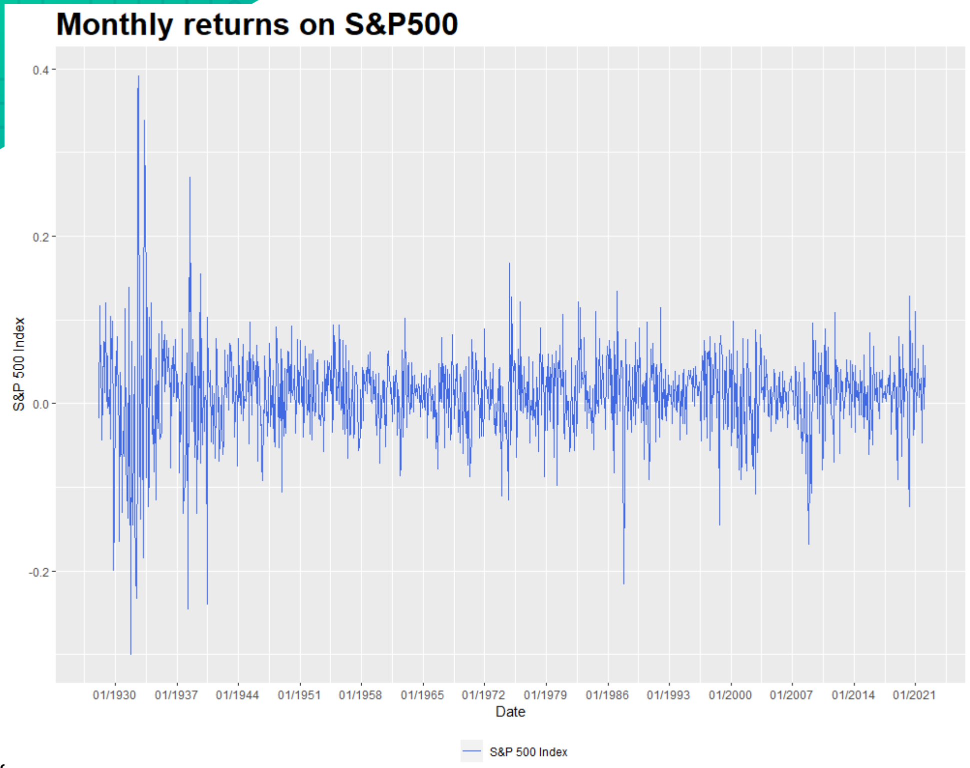
Monthly logreturns on S&P500 Index



Characteristics:

- Non-constant and stochastic volatility
- Volatility clustering, i.e. one observes periods of both low and high volatility
- Leverage effect, i.e. high volatility coincides with negative returns

- If we assume that log-returns come from t-distribution, we will miss the two last characteristics
- Solution (way outside this course): model volatility over time as a mean-reverting model

Class

Distributions with non-negative values

“Book, chapter 2, section 2.5

Gamma distribution with parameters α and β $\Gamma(\alpha, \beta)$

- Very important and showing up in many places
- Density function: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x)$
- $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) \; dx$ – generalisation of the factorial
- Expected value: $\frac{\alpha}{\beta}$
- Variance: $\frac{\alpha}{\beta^2}$
- Excess kurtosis $\frac{6}{\alpha}$
- In R: pgamma, rgamma, dgamma, qgamma

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Properties

- Let $X \sim \Gamma(\alpha_1, \beta)$ and Y$\sim \Gamma(\alpha_2, \beta)$ are two independent random variables then
$$X + Y \sim \Gamma(\alpha_1 + \alpha_2, \beta)$$
- Let $X \sim \Gamma(\alpha, \beta)$ and $c > 0$, then $cX \sim \Gamma(\alpha, \beta / c)$
- If $\alpha = 1$ we get an exponential distribution, which is very important.
- CDF can be nasty in general through so-called incomplete gamma functions.
- Recommendation to use the ready-programmed samplers.

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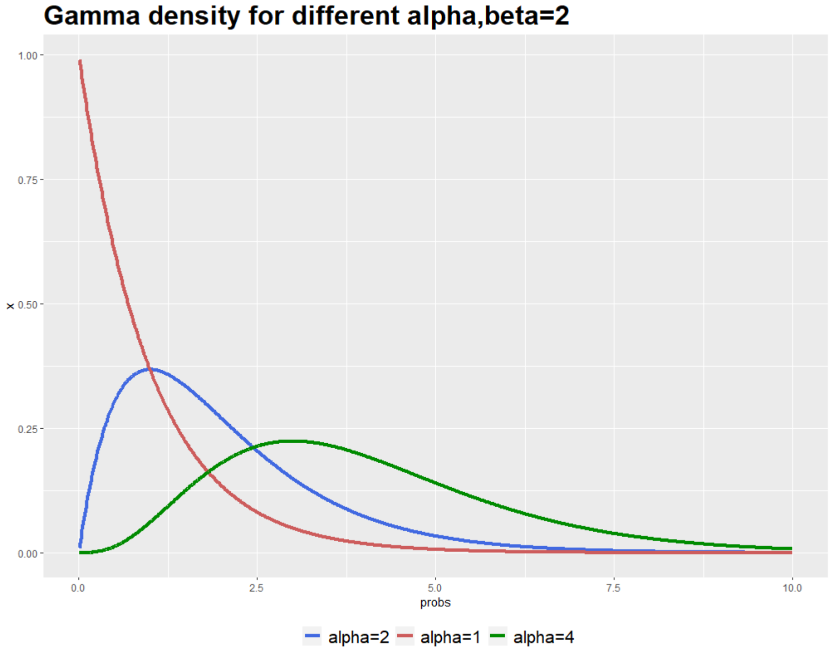
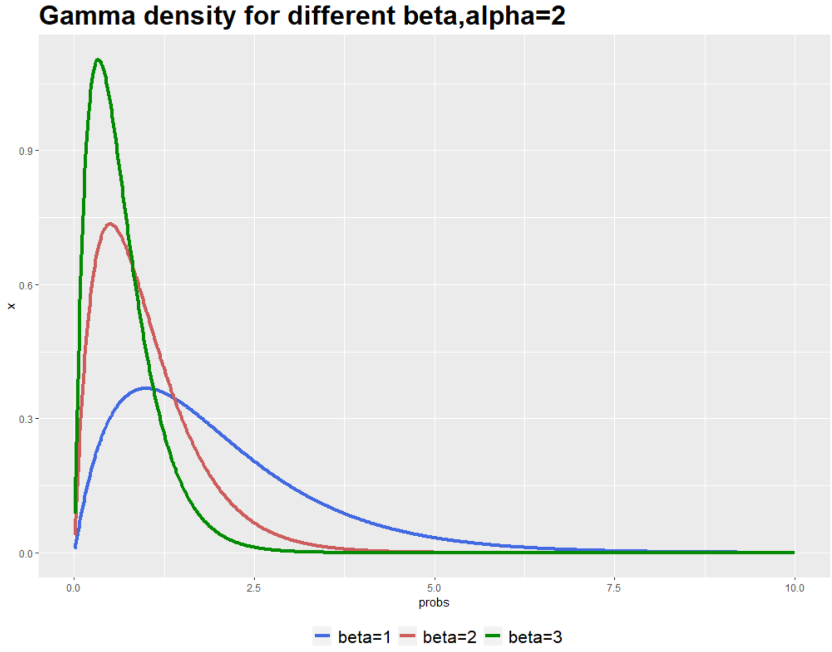
Effect of the parameter change





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Special case of Gamma: exponential

- If $\alpha = 1$ then Gamma distribution turns into exponential distribution
- Density function: $f(x) = \beta \exp(-\beta x)$
- $CDF\ F(x) = 1 - \exp(-\beta x)$
- Expected value: $\frac{1}{\beta}$
- Variance: $\frac{1}{\beta^2}$
- Excess kurtosis: 6
- In R: pexp,dexp,qexp,rexp
- Important property: **memoryless:** $P(X > a + b | X > b) = P(X > a)$

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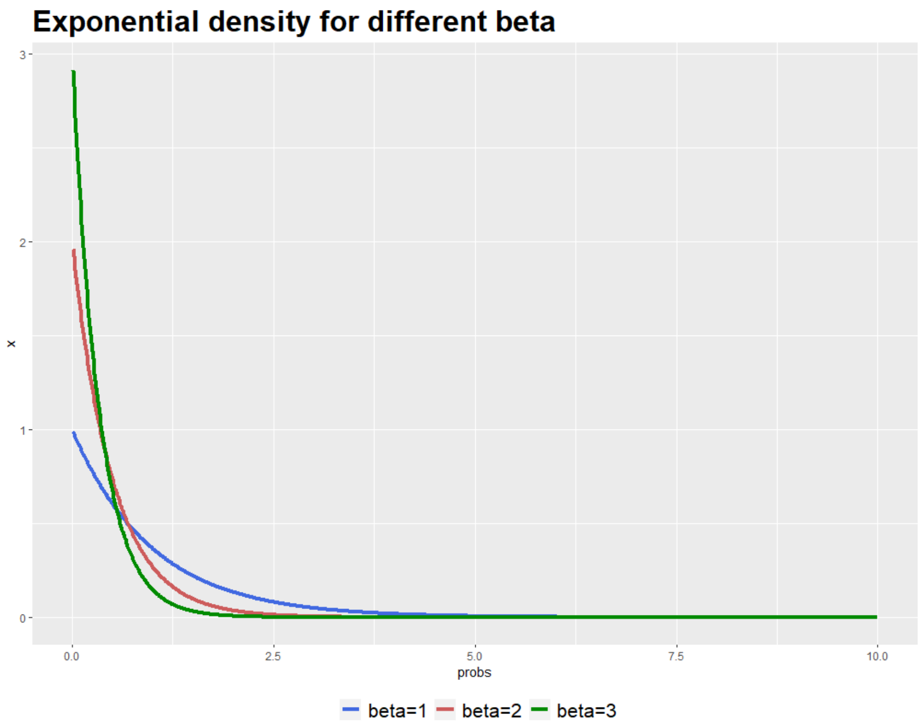
Effect of the parameter change



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Weibull distribution

- If $Y \sim Exp(1)$ then $X = \beta Y^{\frac{1}{\alpha}}$ is Weibull distributed
- Density function $f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$
- CDF $\mathrm{F}(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$
- $E(X) = \beta\Gamma\left(1 + \frac{1}{\alpha}\right)$
- $\mathrm{var}(X) = \beta^2(\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right)^2)$
- In R: dweibull, rweibull, pweibull, qweibull

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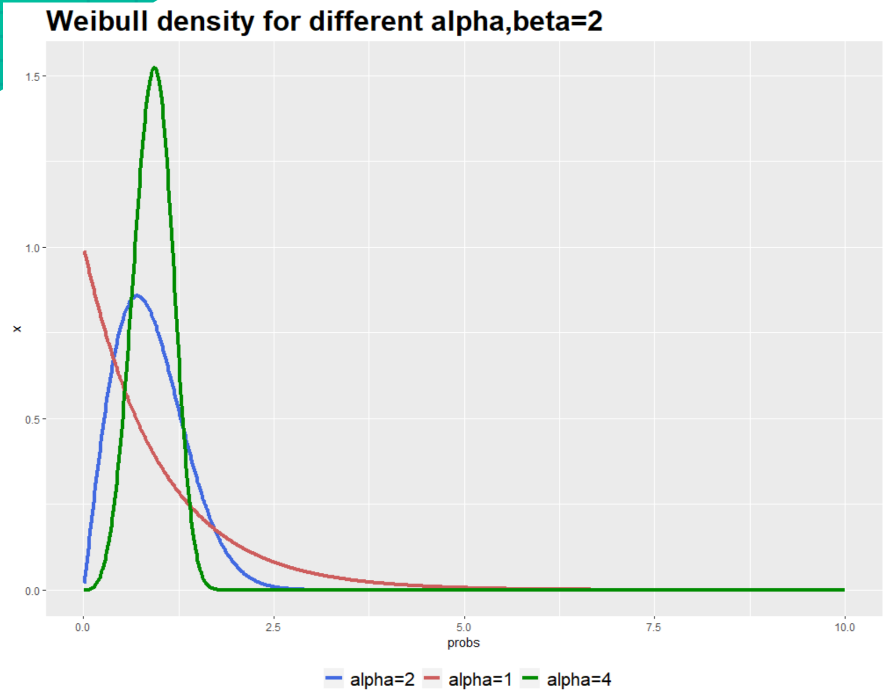
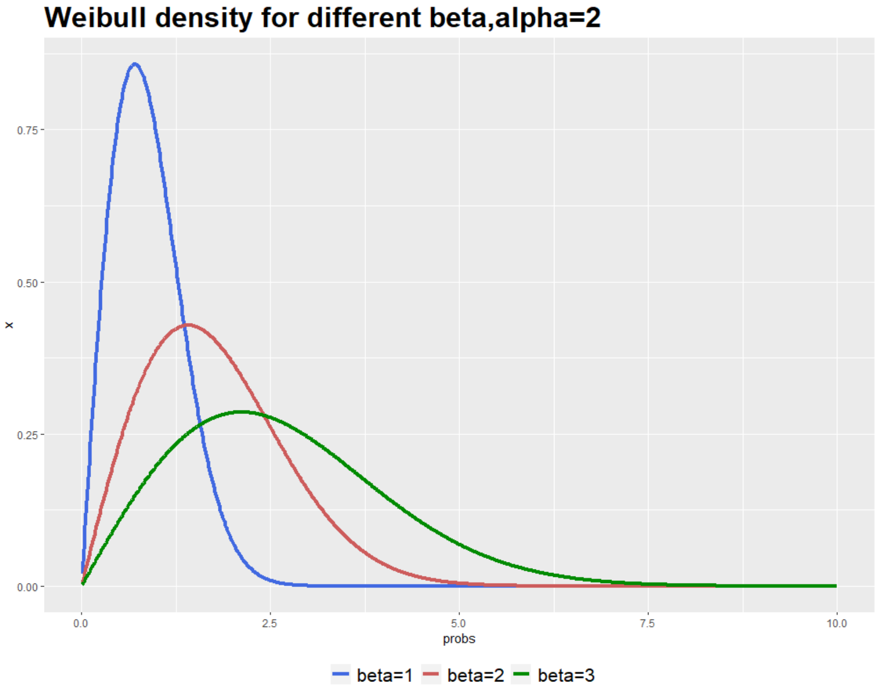
Effect of the parameter change





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HEAVY tails: Pareto distribution

- Density function $f(x) = \frac{\alpha/\beta}{\left(1+\frac{x}{\beta}\right)^{1+\alpha}}$
- CDF $\mathrm{F}(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}$
- $E(X) = \frac{\beta}{\alpha-1}$
- $\mathrm{var}(X) = \frac{E(X)\alpha}{\alpha-2}$
- Sampling through inverse method, or dpareto, ppareto, qpareto,rpareto from EnvStats – note different parametrization

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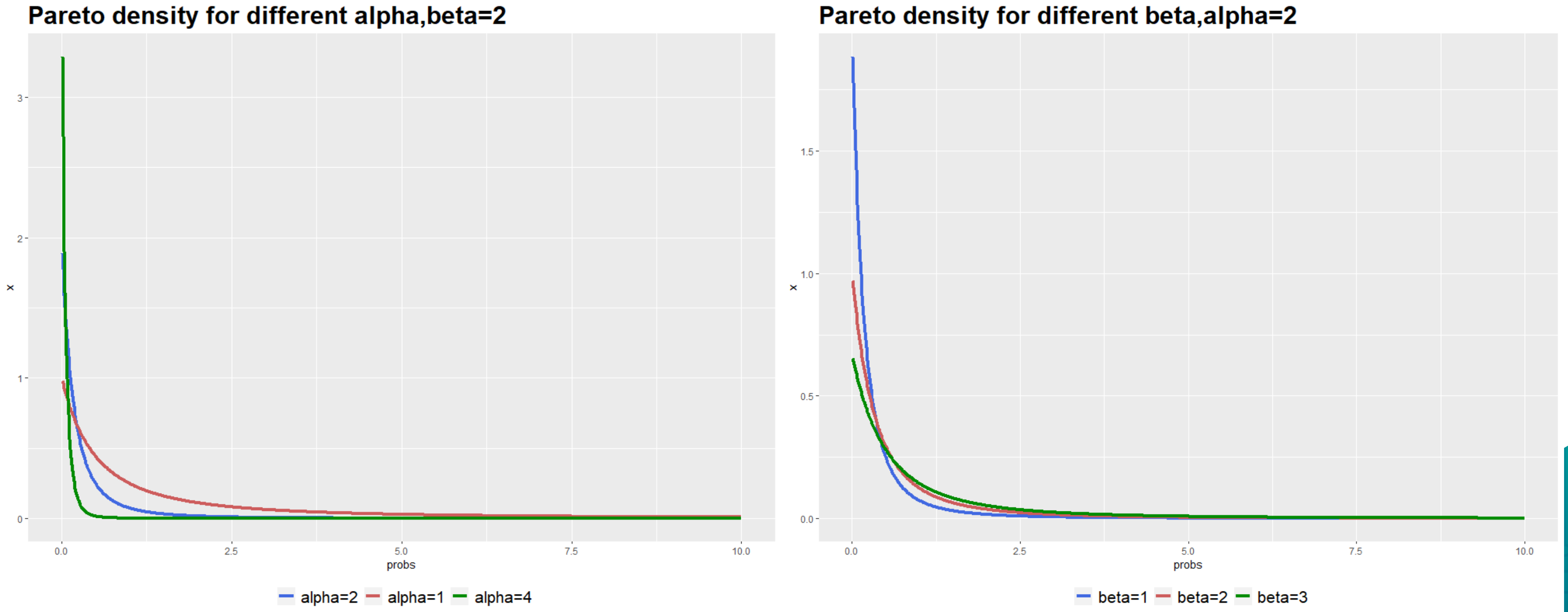
Effect of the parameter change



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Discreet distribution: Poisson

- $X \sim Poiss(\lambda)$ then
- $P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$, k =0,1,2,...
- $E(X) = var(X) = \lambda$
- Properties: a limit distribution of Bernoulli distribution with n trials and $p_n = \frac{\lambda}{n}$.
- Therefore law of rare events: $\text{Bernoulli}(n, p)\ is\ close\ to\ Poiss(np)$ for very large n and very small p
- Relation with exponential: If $Y_1, Y_2, \ldots, Y_n, \ldots \sim Exp(\lambda)\ i.i.d., X \sim Poiss(\lambda t)$ then
- $P(X = k) = P(\sum_{i=1}^{k} Y_i \leq t < \sum_{i=1}^{k+1} Y_i)$
- Used to model number of claims in the non-life insurance

Classified: General Business

Discreet distribution: Negative Binomial

- Derived through Bernoulli trials
- Models the number of failures in a sequence of i.i.d. Bernoulli trials before a specified (non-random) number of successes (denoted r) occurs
- $X \sim NB(r, p)$ then $P(X = k) = \binom{k + r - 1}{k} (1 - \mathrm{p})^{\mathrm{k}} \mathrm{p}^{\mathrm{r}}$
- $E(X) = \frac{pr}{1-p}$
- $\text{var}(X) = \frac{pr}{(1-p)^2}$
- Used to model number of claims in the non-life insurance

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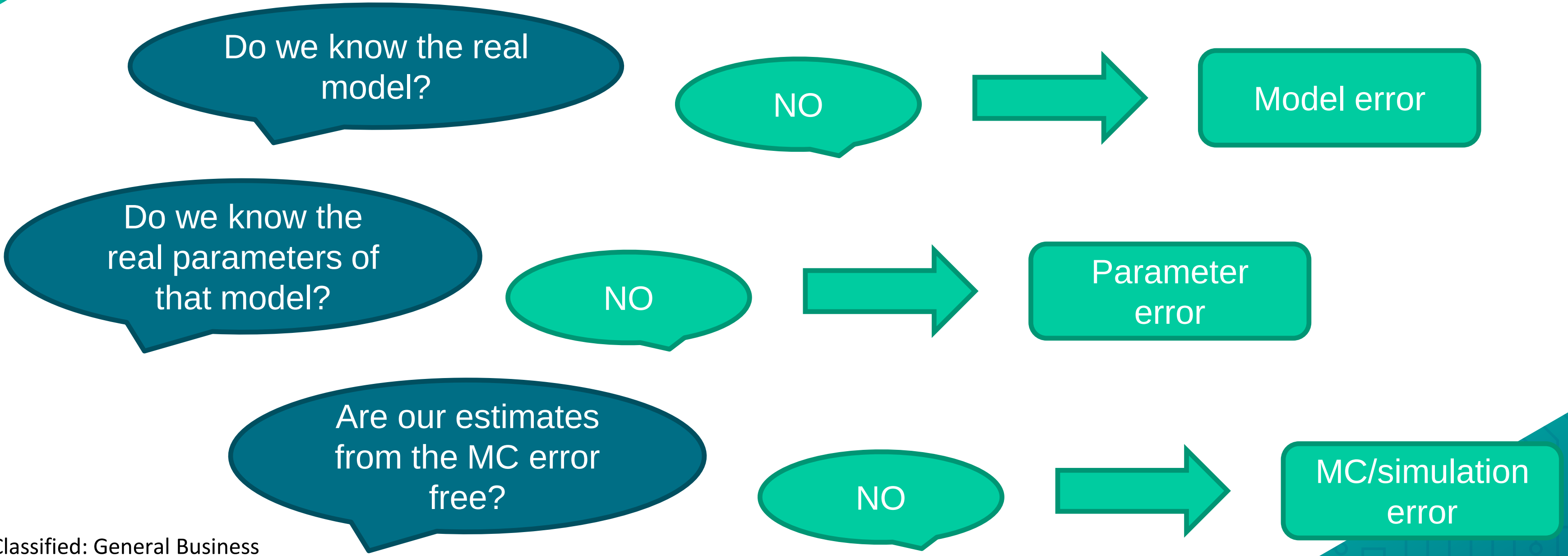
On estimation of the parameters and errors

“ Book, chapter 7, section 7.1-7-3

From historical data to MC

- So far we assumed that we fully know the model and we need to just simulate it.
- Three problems from modelling perspective:



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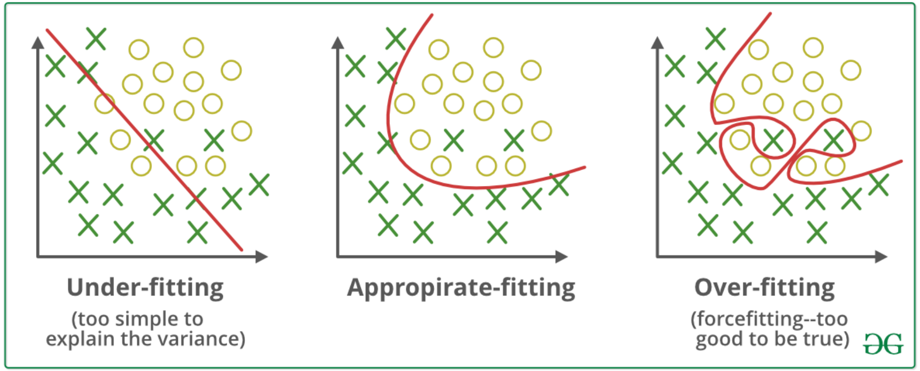
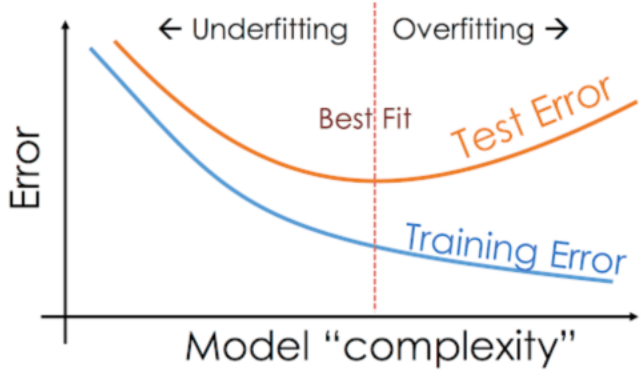
Do we know the model?

- Model/distribution selection is based both on the theory as well as observation of the properties of the data – histograms, Q-Q-plots, quantiles, tail behaviour etc…
- If it is a predictive model (e.g. regression, ML) more complicated – we need to find a balance between trying to reduce the error with which the model predicts the historical values on the **training set**, without making the model overcomplicated and trying to estimate "quirks" in the data. The need for testing on the **validation set**
- Model error (i.e. error due to using the wrong model) most difficult to quantify





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Do we know the parameters?

- Parameter estimation necessary.
- Many ways of estimating parameters:

  a. moment method (for distributions)

  b. maximum-likelihood estimators (for both distributions and some predictive models like regressions)

  c. optimizing the loss function (esp. for machine learning)
- For us moment matching and maximum-likelihood estimation most relevant as we leave predictive models mostly aside in this course.

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Moment method

Assume we have a model based on a distribution, for which we have an explicit formula for the moments, like mean, standard deviation, skew etc. given the parameters $\theta = (\theta_1, \dots, \theta_n)$. E.g. for n=2

$$E(X) = f_1(\theta_1, \theta_2)$$
$$E\big(X - E(X)\big)^2 = f_2(\theta_1, \theta_2) \quad etc.$$

We assume that n (i.e. number of parameters) is less than the number of moments we have an explicit formula for.
We draw a sample and compute the empirical moments, e.g. $\bar{x}$ or s^2.
Then we try to find the solution of the equations which match the function describing the moment with its empirical equivalent, e.g.

$$f_1(\theta_1, \theta_2) = \bar{x}$$
$$f_2(\theta_1, \theta_2) = s^2 \quad etc.$$

Works mostly for relatively easy distributions with max. 2-3 parameters.
Does not guarantee good statistical properties of the estimators which we get:

- estimators consistent but often biased.

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Maximum-likelihood estimators

If we have a random i.i.d. sample $X_1, X_2, \dots, X_n$ coming from a distribution with density f_θ (θ is a vector of unknown parameters), the density of the sample is

$$f_{n,\theta}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_\theta(x_i) \,.$$

Given the random sample $X_1, X_2, \dots, X_n$, it is not outrageous to assume that a good estimate for the parameters is a value $\hat{\theta}$, such that $\hat{\theta}$ maximizes $f_{n,\theta}(X_1, X_2, \dots, X_n)$, i.e. it maximizes the likelihood that the random sample comes from that distribution with that set of parameters.

The method also works if sample not independent, but then no factorization of densities.

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MLE estimation in practise

1. Define likelihood function $L(\theta) = f_{n,\theta}(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} f_\theta(X_i)$.
2. Take ln to be able to operate more easily: $\mathcal{L}(\theta) \coloneqq \ln L(\theta) = \sum_{i=1}^{n} \ln(f_\theta(X_i))$.
3. Maximize the function $\theta \to \mathcal{L}(\theta)$, e.g. by derivation.
4. If maximized by derivation, check that you have a global maximum.

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Example: normal distribution with unknown mean

$$\mathcal{L}(\mu) = \sum_{i=1}^{n} \ln \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} = n \ln \frac{1}{\sqrt{2\pi\sigma}} + \sum_{i=1}^{n} -\frac{(X_i-\mu)^2}{2\sigma^2}$$

$$\frac{d}{d\mu}\mathcal{L}(\mu) = \sum_{i=1}^{n} -(-1) * \frac{2(X_i-\mu)}{2\sigma^2} = \frac{-n\mu + \sum_{i=1}^{n} X_i}{\sigma^2}$$

$$\frac{d}{d\mu}\mathcal{L}(\mu) = 0 \iff \mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\frac{d^2}{d\mu^2}\mathcal{L}(\mu_{MLE}) = -\frac{n}{\sigma^2} < 0$$

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Properties of MLE

- Estimators consistent but can be biased
- Estimators are efficient, i.e. with lowest possible variance
- Parameters not out of "boundaries/domain" which can be the case with moment estimators

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Parameter error – how to measure it

- Use closed form formulas for standard deviation of the estimator
- If you don't have this formula but have really a lot of data – divide the sample in smaller under-samples and estimate parameters from the undersamples

$$X_1, \dots, X_{1000}$$
$$X_{1001}, \dots, X_{2000}$$
$$\dots$$
$$X_{99001}, \dots, X_{100000}$$

$$X_1, \dots, X_{1000}$$

$$X_{1001}, \dots, X_{2000}$$

$$\dots$$

$$X_{99001}, \dots, X_{100000}$$

- But in most cases: better to use all the sample to estimate the parameter.
- How to deal with the parameter error then?

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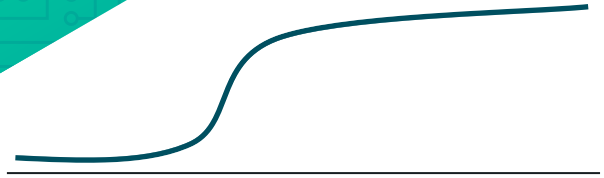
Bootstrap – not included in the curriculum

Idea from a ridiculous story of baron von Münchhausen. In the story baron rode a horse and got stuck in a swamp. To get out he pulled himself out of a swamp by pulling the straps by his boots.

The idea in statistics sounds also pretty ridiculous (at least at first): We are in trouble with too few datapoints to estimate the distribution of the parameter.
So lets treat our SAMPLE as POPULATION and draw i.i.d. sample of size n from our SAMPLE (i.e. resample the SAMPLE).

Bootstrap

Population with CDF F



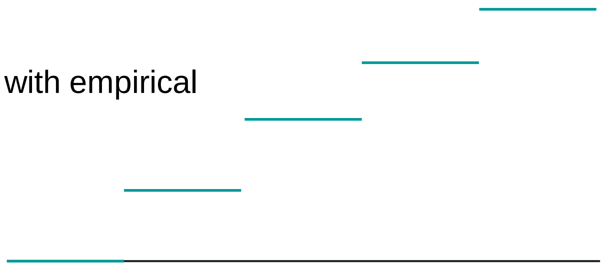
Drawing from F

Sample: $X_1, \ldots, X_{1000}$

Parameter estimation

$\hat{\theta}$

Sample with empirical CDF $\hat{F}_n$



BOOTSTRAP:
redrawing the sample, equivalent to drawing i.i.d. from $\hat{F}_n$

New sample: $Y_1^1, \ldots, Y_n^1$ → $\widehat{\theta_B^1}$

New sample: $Y_1^2, \ldots, Y_n^2$ → $\widehat{\theta_B^2}$

New sample: $Y_1^n, \ldots, Y_n^n$ → $\widehat{\theta_B^n}$

We can do resample as many times as we want (though number of resampling = number of observations in the sample the most typical). Then we can estimate our parameters from those new samples and calculate variance of the estimator, build its confidence intervals etc.

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Why bootstrapping work?

Under very mild assumptions $\hat{F}_n$ converges to real F uniformly in x and with exponential speed.

Therefore if sample is large enough parameters estimated based on $\hat{F}_n$ are not far from parameters estimated based on F and:

$\widehat{\theta_B} - \hat{\theta} \xrightarrow{d} \hat{\theta} - \theta$, where

$\widehat{\theta_B}$ - estimate from the bootstrapped sample

$\hat{\theta}$ - estimate from the empirical sample

θ – true value of the parameter

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When does bootstrap fail

Minimum/maximum

Distributions with infinite variance

Relatively small samples

Extreme quantiles if not enough data

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Non-life insurance

Compound Poisson model

Classified: General Business

Book: Chapter 3, section 3.1, 3.2.1, 3.2.3, 3.2.4, 3.3 (without 3.3.4),

Chapter 6, section 6.1, 6.2.1-6.2.2, 6.2.4, 6.3.1, 6.3.2, 6.3.4

Non–life insurance basics

sharing the risk across many individuals in order to be able to cover unlikely but substantial losses

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Formalizing the model

Assume the following:

- J number of policies, all entered at the same time t=0
- Conditions of the policies either the same for all (*homogenous portfolio* or *homogenous risk group)* or different (*heterogenous risk group)* and the coverage time T is the same (assume T=one year)
- Premiums and *expenses* (i.e. all costs of the insurance company connected to operations, like salaries, IT-systems, commissions to insurance brokers) are known in advance at time t=0.
- The claims are independent and (if homogenous portfolio) identically distributed
- The occurrence of a claim is independent of the claim size and if homogenous portfolio, the distribution og number of claims per policy is identical for all policies
- Policies do not lapse (i.e. insured do not terminate their policies before the end of the coverage time)

Then we can write the underwriting/technical/insurance result of an insurance company as:

- $Result = Premiums - Claims - Expenses$
- How to represent claims given those assumptions?

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Policy level vs. portfolio level

We can either look at the claims arising from one single policy or we can look at the losses from the whole portfolio

Portfolio level

- $Claims = \sum_{i=1}^{N} Z_i$, where
- $\boldsymbol{N}$ – is a random variable describing number of claims in the portfolio
- Z_i - is a size of i-th claim.
- Z_i is independent of N

Policy level – policy j (j in 1,…,J)

- $Claims_j = \sum_{i=1}^{N_j} Z_i^j$, where
- N_j – is a random variable describing number of claims for the policy j
- Z_i^j - is a size of i-th claim.
- Z_i^j is independent of N
- Useful for heterogenous portfolios

One can of course go back to the portfolio level by summing all $Claims_j$

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Stochastic modelling of claims

So far we decided that our claims can be modelled as a random sum of random individual claim amounts. What distributions to choose?

- We focus on Poisson distribution for claim number. Other possible choices: negative binomial
- For claim severirty, we will use distributions from the previous lectures: gamma, pareto, Weibull, lognormal etc.
- Parameters of the Poisson distribution:
  - Let μ be an intensity of claims occuring (i.e. how many claims one expects from one policy)
  - For a single policy j, we use parameter $\lambda = \mu_j T$, where T is the horizon of the policy (usually 1 year)
  - For a homogenous portfolio, we use parameter $\lambda = \mu T J$, J-the number of policies in the portfolio

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Simulating claims – algorithm for homogenous portfolio

1. INPUTS:
   a. T, μ, J # parameters to help define Poisson
   b. Distribution_sampler # a function which generates a random number from the chosen claim severity distribution, given parameter vector θ
   c. θ # vector with parameters
   d. m # number of simulations
2. Set X = vector of 0's of length m #we will store the values in this vector
3. For i in (1 to m) do
   a. Draw N from Poisson distribution with parameter $\lambda = \mu T J$
   b. For k in (1 to N) do
      I. Draw Y from Distribution_sampler(θ)
      II. X[i] := X[i]+Y
4. OUTPUT: return X

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Simulating claims – algorithm for heterogenous portfolio

1. INPUTS:
   a. T, J # parameters to help define Poisson
   b. $\mu = (\mu_j)_{j=1,..,J}$ #vector of intensities for each policy
   c. Distribution_sampler # a function which generates a random number from the chosen claim severity distribution, given parameter vector θ
   d. $\theta = (\theta_j)_{j=1,..,J}$ # vector of parameters for each policy
   e. m # number of simulations
2. Set X = vector of 0's of length m #we will store the values in this vector
3. For i in (1 to m) do
   a. For j in (1 to J) do
      I. Draw N from Poisson distribution with parameter $\lambda = \mu_j T$
      II. For k in (1 to N) do
         i. Draw Y from Distribution_sampler(θ_j)
         ii. X[i] := X[i]+Y
4. OUTPUT: return X

Classified: General Business

Properities

- Let Claims $= X = \sum_{i=1}^{N} Z_i$, where N random variable with values 0,1,2,...
  and Z_i $i.i.d.$, independent of N
- $E(X) = E[Z_i]\, E[N]$
- $\text{Var}(X) = E(N)Var(Z_i) + Var(N)(EZ_i)^2$

- If $N \sim Poiss(\lambda)$ then
- $E(X) = E[Z_i]\, \lambda$
- $\text{Var}(X) = \lambda(Var(Z_i) + (EZ_i)^2)$

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Proof: via conditional expectation

Conditional expectation. Let X, Y – two random variables, and Y is a discrete random variable taking values $\{y_1, y_2, \dots\}$. Then conditional expectation $E(X|Y)$ is a random variable defined as $E(X|Y)(\omega) = \sum_{i=1}^{\infty} E(X|Y = y_i)\, 1_{\{Y=y_i\}}(\omega)$, *where:*

$$E(X|Y = y) = \frac{E\left(X 1_{\{Y=y\}}\right)}{P(Y=y)},\ 1_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}.$$

If X is discrete too, then we have $E(X|Y = y) = \frac{\sum_{i,j=1}^{\infty} x_i 1_{\{y_j=y\}} P(X=x_i, Y=y_j)}{P(Y=y)} = \frac{\sum_i^{\infty} x_i P(X=x_i, Y=y)}{P(Y=y)} = \sum_i^{\infty} x_i P(X = x_i | Y = y)$.

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Properties of conditional expectation

$E\big(E(X|Y)\big) = E(X)$. **(rule of double expectation)**

Proof: $E(X) = E\left[\sum_{i=1}^{\infty} X1_{\{Y=y_i\}}\right] = \sum_{i=1}^{\infty} E\left[X1_{\{Y=y_i\}}\right] = \sum_{i=1}^{\infty} \frac{E\left[X1_{\{Y=y_i\}}\right]}{P(Y=y_i)} P(Y=y_i) = E\big(E(X|Y)\big)$

If X and Y are independent then $E(X|Y) = E(X)$.

Proof: $E(X|Y=y) = \frac{E\left(X1_{\{Y=y\}}\right)}{P(Y=y)} = \frac{E(X)E\left(1_{\{Y=y\}}\right)}{P(Y=y)} = \frac{E(X)P(Y=y)}{P(Y=y)} = E(X)$.

$E(f(Y)X|Y) = f(Y)E(X|Y)$

Proof:$E(f(Y)X|Y=y) = \frac{E\left(f(Y)X1_{\{Y=y\}}\right)}{P(Y=y)} = \frac{E\left(f(y)X1_{\{Y=y\}}\right)}{P(Y=y)} = f(y)\frac{E\left(X1_{\{Y=y\}}\right)}{P(Y=y)} = f(y)E(X|Y=y)$. Therefore:

$E(f(Y)X|Y) = \sum_{i=1}^{\infty} E(f(Y)X|Y=y_i)\,1_{\{Y=y_i\}} = \sum_{i=1}^{\infty} f(y_i)E(X|Y=y_i)1_{\{Y=y_i\}} =$

$\sum_{i=1}^{\infty} f(y_i)1_{\{Y=y_i\}} \sum_{j=1}^{\infty} E\big(X|Y=y_j\big)1_{\{Y=y_j\}} = f(Y)E(X|Y)$.

Similarly, one can prove that $E(E(X|Y)|Y) = E(X|Y)$.

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Properties of conditional expectation

$Var(X) = Var\big(E(X|Y)\big) + E(Var(X|Y))$. **(rule of double variance)**

Proof: $Var(X) = E\big(X - E(X)\big)^2 = E\left[E\left[\big(X - E(X)\big)^2|Y\right]\right] = E\left[E\left[\big(X - E(X|Y) + E(X|Y) - E(X)\big)^2|Y\right]\right] =$

$$E\left[E\left[\big(X - E(X|Y)\big)^2 + \big(E(X|Y) - E(X)\big)^2 - 2\big(X - E(X|Y)\big)\big(E(X|Y) - E(X)\big)|Y\right]\right] =$$

$$E\left[E\left[\big(X - E(X|Y)\big)^2|Y\right]\right] + E\left[E\left[\big(E(X|Y) - E(X)\big)^2|Y\right]\right]$$

$$- 2E\left[E[\big(X - E(X|Y)\big)\big(E(X|Y) - E(X)\big)|Y]\right] =$$

$$E\big(Var(X|Y)\big) + E\left[E\left[\big(E(X|Y) - E(E(X|Y))\big)^2|Y\right]\right] \text{-}2E\left[\big(E(X|Y) - E(X)\big)E[\big(X - E(X|Y)\big)|Y]\right]=$$

$$E\big(Var(X|Y)\big) + E\left[\big(E(X|Y) - E(E(X|Y))\big)^2\right] - 2E[\big(E(X|Y) - E(X)\big)\big(E(X|Y) - E(X|Y)\big)]$$

$$= E\big(Var(X|Y)\big) + Var(E(X|Y)$$

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Back to compound Poisson model

$$E(X|N=n) = E\left[\sum_{i=1}^{N} Z_i \,|N=n\right] = \frac{E[\sum_{i=1}^{N} Z_i 1_{\{N=n\}}]}{P(N=n)} = \frac{E\left[\sum_{i=1}^{n} Z_i \, 1_{\{N=n\}}\right]}{P(N=n)} = \frac{E[\sum_{i=1}^{n} Z_i] E\left[1_{\{N=n\}}\right]}{P(N=n)}$$

$$= E\left[\sum_{i=1}^{n} Z_i\right] \frac{P(N=n)}{P(N=n)} = nE(Z_1).$$

Therefore: $E(X|N) = NE(Z_1)$

Similarly:

$Var(X|N=n) = Var\left[\sum_{i=1}^{N} Z_i \,|N=n\right] = nVar(Z_1)$ and $Var(X|N) = NVar(Z_1)$

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Coming back to the proof of properties

$E(X) = E[Z_i]\, E[N]$

Proof: by rule of double expectation:

$E(X) = E[E(X|N)] = E\big(N * E(Z_1)\big) = E(N) * E(Z_1).$

$Var(X) = E(N)Var(Z_i) + Var(N)(EZ_i)^2$

Proof by rule of double variance:

$Var(X) = Var\big(E(X|Y)\big) + E\big(Var(X|Y)\big) = Var\big(N * E(Z_1)\big) + E\big(N * Var(Z_1)\big)$

$= Var(N) * E(Z_1)^2 + E(N) * Var(Z_1)$

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Properties – proofs without conditional expectation

- Let $Claims = X = \sum_{i=1}^{N} Z_i$, where N counting variable and Z_i $i.i.d.$, independent of N
- $E(X) = E[\sum_{i=1}^{N} Z_i] = E[\sum_{k=0}^{\infty} 1_{N=k} \sum_{i=1}^{N} Z_i] = \sum_{k=0}^{\infty} E[1_{N=k} \sum_{i=1}^{k} Z_i] = \sum_{k=0}^{\infty} P(N = k) E[\sum_{i=1}^{k} Z_i] =$
  $= \sum_{k=0}^{\infty} P(N = k) k \; E[Z_i] = E[Z_i] \, E[N]$
- $\text{Var}(X) = E\left[\left(\sum_{i=1}^{N} Z_i - E(\sum_{i=1}^{N} Z_i)\right)^2\right] = E\left[\sum_{k=0}^{\infty} 1_{N=k} \left(\sum_{i=1}^{N} Z_i - E(\sum_{i=1}^{N} Z_i)\right)^2\right] =$

  $E\left[\sum_{k=0}^{\infty} 1_{N=k} \left(\sum_{i=1}^{k} Z_i - E(\sum_{i=1}^{N} Z_i)\right)^2\right] = E\left[\sum_{k=0}^{\infty} 1_{N=k} \left(\sum_{i=1}^{k} Z_i - E(\sum_{i=1}^{k} Z_i) + E(\sum_{i=1}^{k} Z_i) - E(\sum_{i=1}^{N} Z_i)\right)^2\right] =$

  $\sum_{k=0}^{\infty} P(N = k) E\left[\left(\sum_{i=1}^{k} Z_i - E(\sum_{i=1}^{k} Z_i)\right)^2 + \left(E(\sum_{i=1}^{k} Z_i) - E(\sum_{i=1}^{N} Z_i)\right)^2 +\right.$

  $\left. 2\left(\sum_{i=1}^{k} Z_i - E(\sum_{i=1}^{k} Z_i)\right)\left(E(\sum_{i=1}^{k} Z_i) - E(\sum_{i=1}^{N} Z_i)\right)\right] =$
- $\sum_{k=0}^{\infty} P(N = k) E\left[\left(\sum_{i=1}^{k} Z_i - E(\sum_{i=1}^{k} Z_i)\right)^2\right] + \sum_{k=0}^{\infty} P(N = k) E\left[\left(E(\sum_{i=1}^{k} Z_i) - E(\sum_{i=1}^{N} Z_i)\right)^2\right] + 0 =$
- $\sum_{k=0}^{\infty} P(N = k) k \, Var(Z_i) + \sum_{k=0}^{\infty} P(N = k)(kE(Z_i) - E(N)E(Z_i))^2 =$
- $E(N)Var(Z_i) + \sum_{k=0}^{\infty} P(N = k)\left(k - E(N)\right)^2 (EZ_i)^2 = E(N)Var(Z_i) + Var(N)(EZ_i)^2$

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Application to Compound Poisson

- Let $\lambda = \mu T J$. Consider $\frac{sd(X)}{E(X)}$, i.e. standard deviation of all claims as a percentage of expected claim amount. Then:
- $\mathrm{E(X)} = \mu T J * E(Z_i)$
- $\frac{sd(X)}{\mathrm{E(X)}} = \frac{1}{\mu T J * E(Z_i)} [\mu T J \ (Var(Z_i) + (EZ_i)^2)]^{0.5} = \frac{1}{\sqrt{J}} \frac{\sqrt{(Var(Z_i)+(EZ_i)^2)}}{E(Z_i)\sqrt{\mu T}}$
- Risk per unit of exposure decreases to 0 if J large enough
- Crucial assumption here: $N, Z_1, Z_2, \ldots$ independent!!!!
- In real life assumption is violated and therefore insurance company faces some risks even though large diversified portfolio

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Pure premium

- Pure premium π^{pu} relates to the level of premiums which covers only the expected claim for a policy, without other costs or reward for the insurance company for taking risk
- For single policy: $X = \sum_{i=1}^{N} Z_i$, where N $\sim Poiss(\mu_i T)$, therefore $\pi^{pu} = EX = \mu_i T * E(Z_i)$
- For portfolio: $X = \sum_{i=1}^{N} Z_i$, where N $\sim Poiss(\mu J T)$, therefore

$$\pi^{pu} = \frac{EX}{J} = \mu J T * \frac{E(Z_i)}{J} = \mu T * E(Z_i)$$

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Premium - loading

In real life premiums are usually larger than pure premium. Why?

- Expenses/operating costs – salaries, cost of IT-systems etc.
- Even for large portfolios risk is not completely diversified, so insurance companies should be compensated for it.
- Usually insurance companies are for profit, so the owners/investors require some profit margin

Therefore: real premium π can be decomposed as a pure premium and a loading γ such that

$$\pi = \pi^{pu}(1 + \gamma)$$

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More on claims

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“Book: chapter 3, section 3.2.2, 3.3.5, 3.3.6

Difference between damage and paid out claim – clauses

- Insurance company may cover only a part of damage which is insured. Specifics what is covered and to what degree specified in the terms and conditions
- In book, it is defined via a function H such that Claim=H(Damage).
- Deductible – the part of the damage which is covered by the insured
- Sums insured – maximum loss which is covered by an insurance company
- Exclusions: risks which are not covered at all
- Example:
  - Property insurance: deductible 4000 kr, sums insured 10 000 000 kr, exclusion: "grov uaktsomhet/gross negligence" "war"
    Claim= min(max(Damage – 4000;0);10 000 000)*1_(not gross negligence nor war)
  - Pet insurance (medical expenses): deductible 20% of expenses, but at least 1 500 kr, sums insured 30 000 kr; exclusion: "medical expenses not caused by illness or accident"
    Claim = min(max(Damage– max(20% Damage; 1500);0);30 000) 1_(illness or accident)

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Why deductible, sums insured and exclusions?

Deductible

- Limits **moral hazard**, i.e. that the behaviour on insured changes with insurance
- With deductible an insured still has a financial stake in preventing the damage
- Claim handling costs money, and deductible makes reporting the claim under the limit pointless

Sums insured

- Sets a clear limit for the liability of an insurance company
- E.g. marine insurance and oil spill – can be very expensive (oil tanker Exxon Valdez in 1989)

Exclusions

- Risks which are difficult to insure or not insurable at all
- E.g. war:
  - can causes a lot of large claims for property or life with high probability.
  - impossible to send a *claim adjuster* (takstmann) to assess the damage and construction team to repair it
- Often same exclusions as in the **reinsurance programme**

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Implications for claim modelling

The deductible, sums insured and exclusions must of course be taken into account when modelling claim frequency and severity.
However, modelling in practise usually occurs on the historical claim data, not the damage data. So is it really a problem?

YES, it can be still problematic, mostly via appropriateness of data, e.g.:

1. You keep deductible constant over the years, but the prices changes a lot – Covid 19 and "terrace building frenzy"
2. You have claim sizes after deductible, but the deductible varies across the policies. What is real "damage frequency"?
3. New exclusion has been introduced. You may need to take care to exclude such claims from your dataset (e.g. "silent cyber"-coverage, i.e. coverage of cyber risk damages via general clauses)
4. In property catastrophe modelling, loss is often modelled as % of sums insured. But sums insured very often not updated during the renewal, while cost can increase.

Even though such questions are important in real life, from now on, we assume (if not stated otherwise) that we work on claim data and there are no problems connected to that.

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Difference between a paid put claim and loss for an insurance company - reinsurance

- Insurance company might want to insure some risks, but then move some part of the risk (**cedes the risk**) to another company – **reinsurance company.**
- Reinsurance companies are big global companies which use exactly the same type of argument (LLN) to re-insure and diversify risks which regular reinsurance companies usually do not want to keep
- In practice: Insurance company **cedes** a part of premiums by paying reinsurance premium to a reinsurance company while reinsurance company is going to compensate the insurance companies for all the claims which qualify
- Reinsurance can be either per event or per portfolio, i.e. the reinsurance can take over a part of risk for singe events (e.g. large claims) in a portfolio or a part of general risk in portfolio (e.g. if the portfolio has generated too much loses in total)

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Difference between a paid put claim and loss for an insurance company - reinsurance

- Example of risks which are usually reinsured:
  - Catastrophe risks – both NatCat (natural catastrophes) and Man-Made
  - Non-CAT large claims
- Reinsurance can be also done between companies within a group (internal reinsurance) which can be explained from capital management/optimization perspective
- Internal reinsurance can be also used for tax reasons

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Notation and definitions

Definitions

Risk/premiums ceded – risk or premiums which are moved to reinsurance company

Reinsurance premium = what insurance company pays to the reinsurance company. In practice the same as premiums ceded

Pure reinsurance premium = premium which is set on the level of expected claims covered by reinsurer

Risk retained – the risk left to the insurance company after the reinsurance contact

Reinsurance recoverables – part of claims which the reinsurance company is responsible for

Gross/net claims – claim amounts before/after reinsurance recoverables

Gross/net premiums – premiums before/after deducting reinsurance premiums

Notation

Let $X = \sum_{i=1}^{N} Z_i$ be a sum of claim amounts before reinsurance

Let H^{event} be a function which calculates reinsurance recoverables for a contract per event

Then $X^{re} = \sum_{i=1}^{N} H^{event}(Z_i)$ is a sum of reinsurance recoverables from the contract

Let $H^{portfolio}$ be a function which calculates reinsurance recoverables for a contract per portfolio

Then $X^{re} = H^{porfolio}(\sum_{i=1}^{N} Z_i)$ is a sum of reinsurance recoverables from the contract

Net claims are then denoted as

$$X^{ce} = X - X^{re}$$

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Types of reinsurance

Proportional – e.g. quota share

- An insurance company cedes a predefined percentage of each premium from a specific line of business to the reinsurance company and reinsurance company pays the same percentage of each claim. H^{event}(x)= $H^{portfolio}(x) = \gamma x$
- Why do that?
  - When not having enough capital to write new business
  - Capital optimization within a group – smaller daughter companies less diversified then parent company and therefore requires more capital. Reinsurance can move the risk to the parent.
  - Tax reasons – move the profit to Bermuda ;-)

Non-Proportional

- Many variations, but in principle used to transfer tail risk to the reinsurance company, e.g. on next slide
- Why do that?
  - Protect capital from impact of rare events or cumulation of them
  - Capital optimization - you need to hold less capital to cover events from point above
  - Reduce the volatility of your results

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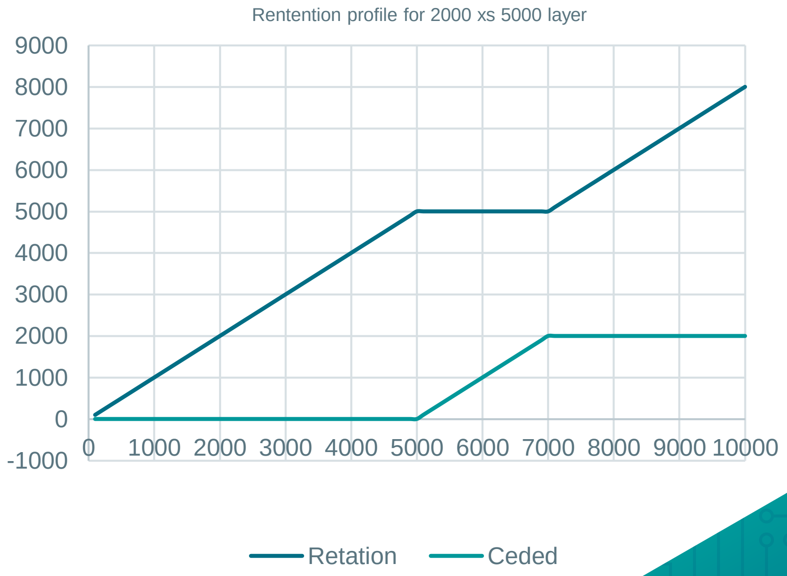
Non-proportional reinsurance – excess of loss

Most generic: excess-of-loss (layer B xs A: notation in the book is opposite) per event (with unlimited reinstatements). Insurance company pays a fixed premium, and reinsurance company pays the amount exceeding A, but up to B from each claim.

Typical to stack up layers from different reinsurers, e.g. have layer 400xs100, then 500xs500 etc.

Example: https://www.igpandi.org/reinsurance - very complicated structure of a Pool of P&I Clubs (i.e. marine insurers)



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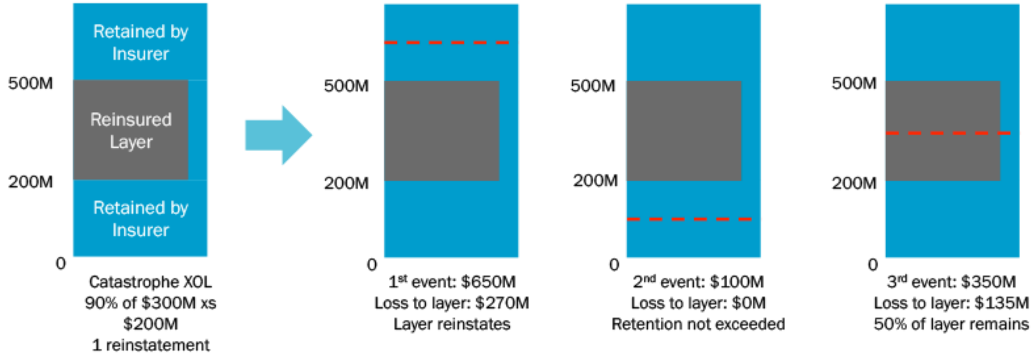
Other examples – reinstatements (not in curriculum)

- Excess of loss with a fixed number reinstatements – you can use the layer fully only the fixed amount of times



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Other example: Stop loss

Example of the contract on the portfolio level, not on the claim/event level. Reinsurer covers all the claims (regardless size) when the total amount of claims so far has exceeded a predefined threshold (usually % of premiums).

Example: assume we have written 100 million kr premiums and we have 110% stop loss contract. Then the insurance will cover all new claims since the moment our cumulative claims reached the level110 million kr.

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Reinsurance per event –what does it mean?

Excess-of-loss mostly written per event/peril. Does it mean per claim?

Not necessarily. One event can cause many claims from many different products:

- Catastrophe like windstorms, earthquakes etc.
- Events like introducing covid-19 restrictions: what with travels cancelled?
- Events like frost waves – here one aggregates all the claims from the 2 week-period and sees if the aggregate amount reaches any layer

Conclusion: in order to model such reinsurance correctly, we need to have a more complex model:

- Most likely we need to model small claims and large /CAT claims separately
- Catastrophe model per peril (e.g. windstorm) which can generate claims from many products
- Desirable with having a "date" for a claim or cluster such claims in appropriate window
- Even then it is only a model: sometimes better not to overcomplicate

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Beyond compound Poisson – not in curriculum

I.e. completely different way of modelling

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Claim type by size – not in the curriculum

Attritional/frequency claims

- Small claims which happen relatively often and are not connected with each other in any meaningful way.
- No need to worry about reinsurance
- Modelling approach: assume underlying compound Poisson model but usually do not simulate it. If you need to simulate, better to model it as an aggregate (e.g. aggregated claim amount via some thick-tailed distribution)

Large claims

- More severe claims which happen relatively rarely, but are mostly not connected with each other in any meaningful way.
- Event = claim, so reinsurance on claim basis
- Modelling approach: assume underlying compound Poisson in most of the cases, including when you need to simulate it.

CAT-events

- Rare events, which may trigger many claims in many lines of business
- Event = peril (windstorm, frost, storm surge, flood, terrorism etc.)
- Modelling approach: either assume underlying compound Poisson or use a specific CAT model

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Another approach to modelling: CAT model – not in the curriculum

Hazard

- Model contains a catalogue of events together with the frequency of the event
- E.g. Windstorm model would have a catalogue of storms which would tell you what was the maximum wind speed at each location
- Events are often generated via complicated physical models (think PDEs and Navier-Stokes equation)

Vulnerability

- This component takes an event from hazard module and the real exposure (e.g. all the insured house, with their detailed location, number of storeys, type of roof) and then simulate losses for the given event and the exposure
- Losses are simulated using so-called vulnerability curves, which link level of peril (e.g. windspeed) to the % loss of sums insured

Financial loss

- This component takes the losses generated in the vulnerability module and applies reinsurance contracts to calculate what reinsurance is going to cover

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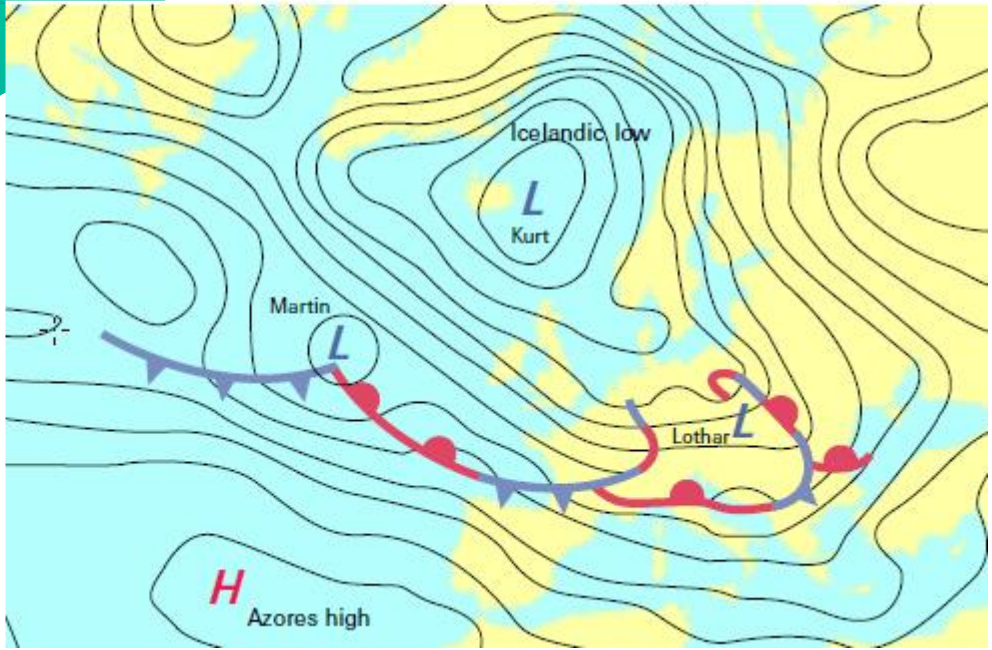
Example: Windstorm, hazard – not in curriculum



- Hazard model usually based on super complicated model of the atmosphere
- Such models are based on PDEs and model e.g. air pressure, temperature, air moisture, energy in the air, the orography (i.e. formation of the terrain)
- Final result: a catalogue containing "maps" with maximum wind speeds at different location per event/windstorm
- It can be very detailed: up to even on 1-3 km-grid

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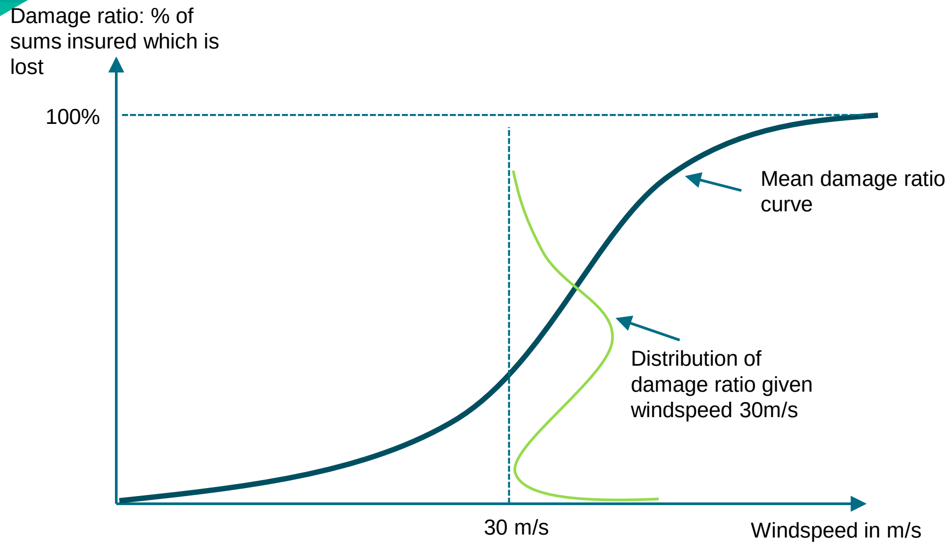
Example: Windstorm, vulnerability – not in curriculum



- Vulnerability modelled via "vulnerability curves"
- The curve can be either constructed using an engineering model (one builds a real model of a house or a computer model and exposes to the peril, here windspeed) or using statistical methods on historical data. Most likely one needs to combine both approaches.
- Such curves are constructed for different types of buildings (commercial, industrial, residential), different construction types (wooden, concrete, steel….), different number of storeys, different types of roof….
- At last, the model takes such a curve for appropriate building, takes the windspeed and simulates the loss given all those characteristics

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Non-life insurance contract over time

I.e. how to set up reserves and update reserves over time

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" In curriculum but not in book

Reserving in the book

In Chapter 1, section 1.2.3. reserves (or ~~solvency capital~~) are defined as $(1-\epsilon)$-quantile of a process $X = \sum_{i=1}^{N} Z_i$, N-random number of claims, Z_i - size of i-th claim.

Such a definition is **wrong** and **confusing** because:

- Does not take into account running of the time, reporting of the claims and payments
- solvency capital is a part of equity/own funds, not a part of reserves
- reserves often consists of a "best estimate", i.e. expected amount, and "risk margin/adjustment" which measures the risk that the full amount paid out is different than the expectation. However neither risk margin nor risk adjustment is defined as percentile of X.

Therefore, I would like you to forget about this definition.

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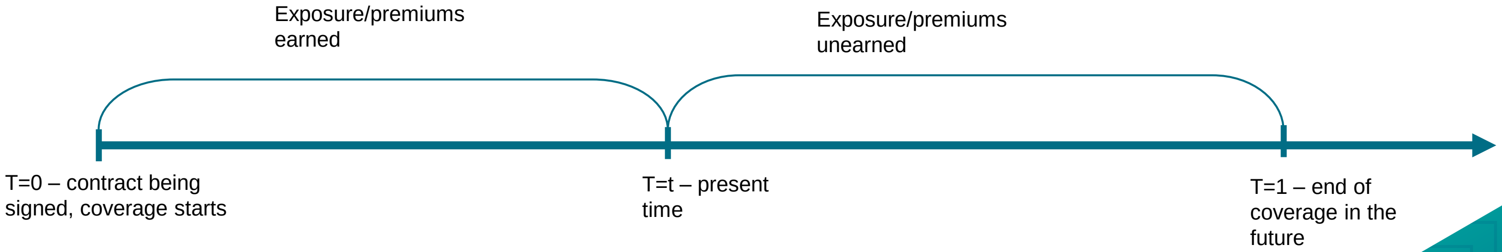
A priori risk vs. real experience

- When pricing, we use the model to analyse the risk during the course of coverage of an insurance product but BEFORE that coverage starts. This risk then is called **a priori** risk (*a priori="from the earlier"* in Latin)
- However, after the contract is signed, and the coverage starts, we get some knowledge about the risk and we use this to set reserves.

- Firstly, we distinguish between **unearned premiums/exposure** and **earned premiums/exposure**
- **Earned exposure/premiums** is associated with the part of coverage which has already happened, i.e. claims which have already occurred. Based on reported claims, we have more information about the what has happened and we can reflect that in our estimates
- **Unearned exposure/premiums** is connected to the part of the coverage period in the future, i.e. future claims. Here we still do not know what is going to happen and a priori model is still what we have



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Calculation of earned/unearned/paid premium – policy level

- In most cases the exposure is earned linearly – i.e. we assume that underlying risks are uniformly distributed w.r.t. time
- Then $earned\ premium = \frac{time\ since\ signing\ the\ contract}{the\ length\ of\ the\ coverage} * overall\ premium$ and $unearned\ premium = overall\ premium\ - earned\ premium$.
- However in some cases better to use other models for earning the exposure (e.g. liability insurance for a ski school in Holmenkollen)
- Note that earned premium is not the same as paid premium. Paid premiums reflects what has been paid to the insurance company, regardless if it has been earned or not.
- Different payment schedules:
    - Upfront one-time premium
    - Monthly paid premiums, quarterly premiums etc.

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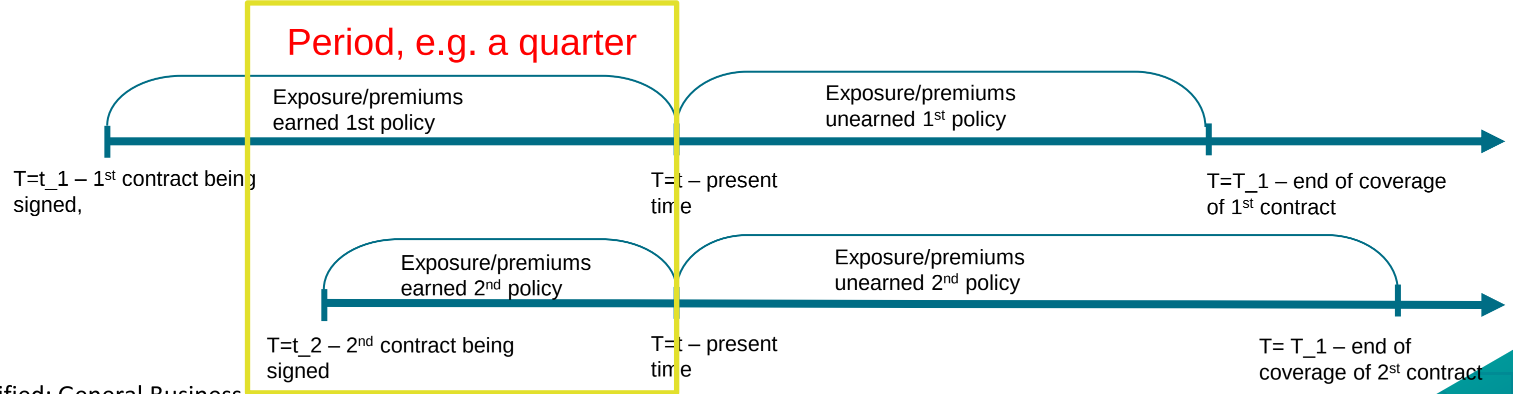
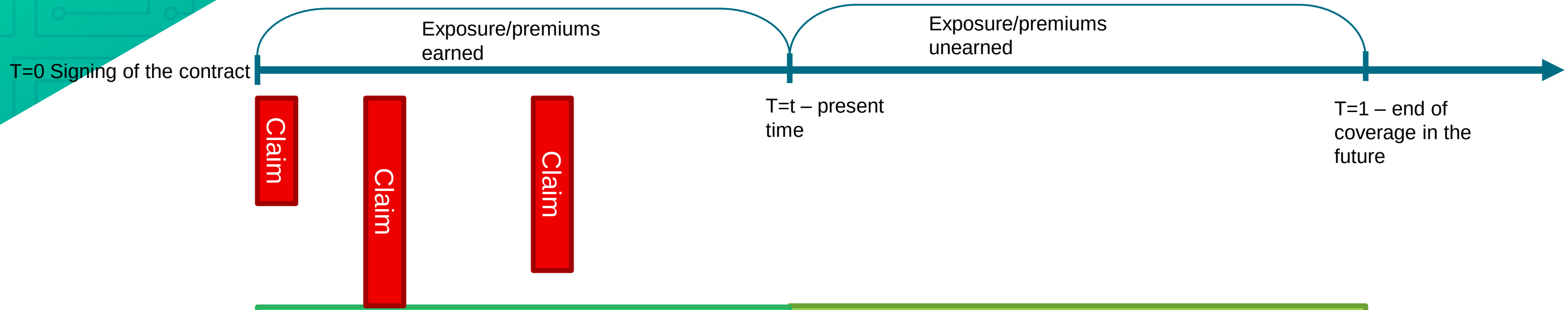
Premiums per portfolio level – some definitions

- **Written premiums** from a period – the sum of all the premium amounts (regardless if they been paid or not, earned or not) from contracts signed in a given period – <u>measure of the size of the company</u>
- **Earned premiums** from a period – the sum of all the premiums which have been earned in the period (regardless if they have been paid or not, nor when the contracts have been signed) – <u>measure for the risk exposure in that period</u>
- Often one additionally adds "gross" or "net" to indicate if the reinsurance premiums have been deducted from the sum, e.g. net earned premiums.



From earning exposure to reserving – policy level



Reserve for earned exposure

Different names (and principles for calculation) for different purposes:

1. Claim provision in Solvency II
2. Liability for incurred claims (IFRS 17, i.e. accounting)

Reserve for un-earned exposure

Different names (and principles for calculation) for different purposes:

1. Premium provision in Solvency II
2. Liability for remaining coverage (IFRS 17, i.e. accounting)

Solvency II – EU regulation which defines e.g. how much capital insurance company must hold, how to calculate that capital, among other things how to calculate reserves

IFRS 17 – new accounting standard which is going to be implemented from January 1st 2023 for listed insurance companies

Some notes

- A if a policy has just been signed, it will have only reserve for un-earned exposure
- If some time passed, there will be reserves for both unearned and earned exposure
- After the end of the coverage, there is no reserve for un-earned exposure, but there still might be a reserve for earned exposure, as long as all the claims have not been settled.

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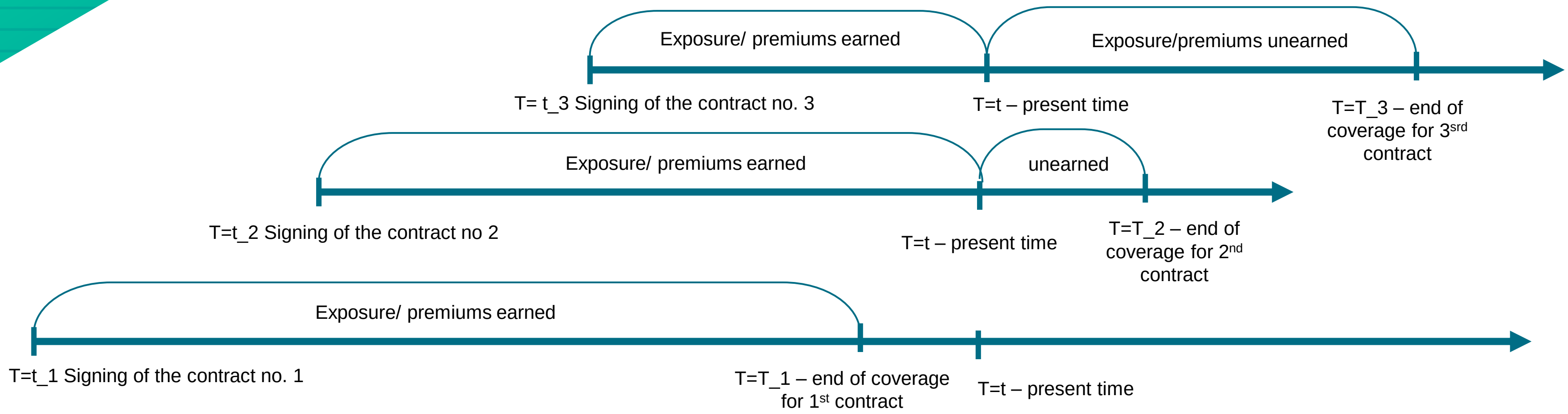
From earning exposure to reserving – Portfolio level



Reserve for earned exposure	Reserve for un-earned exposure

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Reserve for unearned exposure

Solvency II

(Best estimate of) Premium Provision = discounted sum of expected claims and of expected expenses minus discounted sum of expected future premiums connected to unearned exposure from all the active policies

IFRS17

Liability for remaining coverage = unearned premiums minus sum of expected future premiums connected to unearned exposure from all the active policies (minus acquisition costs i.e. commissions to the brokers etc.)

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Example – reserve for unearned exposure, quarterly premium

Assume that at January 1st we signed an insurance contract with premiums 100 kr, with premiums paid either:

1. Upfront for the whole year
2. Quarterly at the end of the quarter,
3. Quarterly, but upfront at the beginning of each quarter.

Assume that exposure is earned linearly and the insurer expects that it is going to pay as claims total 60 kr and as expenses 20 kr. Interest rate is 0% and acquisition expenses are 0.

What is the reserve after a first quarter?

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Example – reserve for unearned exposure, upfront premium

Premium provision per Q1 = 60kr = Discounted expected claims in Q2-Q4 = 75% * 60 kr + Discounted expected expenses in Q2-Q4 = 75% * 20kr − Discounted expected future premiums in Q2-Q4= 0kr

Liabilitiy for remaining coverge = 75 kr = Unearned premium = 75 % * 100 − Acquisition costs = 0kr − Expected future premiums in Q2-Q4= 0kr

Difference is 15 kr – Can anyone give an interpretation to this amount?

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Example – reserve for unearned exposure, quarterly premium paid at the end of the quarter

Premium provision per Q1 = -15kr = Discounted expected claims in Q2-Q4 = 75% * 60 kr + Discounted expected expenses in Q2-Q4 = 75% * 20kr − Discounted expected future premiums in Q2-Q4= 75% * 100kr

Liabilitiy for remaining coverge = 0 kr = Unearned premium = 75 % * 100 − Acquisition costs = 0kr − Expected future premiums in Q2-Q4= 75% * 100kr

The difference from Solvency II still is 15kr.

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Example – reserve for unearned exposure, quarterly premium paid upfront each of the quarter

Premium provision per Q1 = 10kr = Discounted expected claims in Q2-Q4 = 75% * 60 kr + Discounted expected expenses in Q2-Q4 = 75% * 20kr − Discounted expected future premiums in Q2-Q4= 50% * 100kr

Liabilitiy for remaining coverge = 25 kr = Unearned premium = 75 % * 100 − Acquisition costs = 0kr − Expected future premiums in Q2-Q4= 50% * 100kr

The difference from Solvency II still is 15kr.

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How our compound Poisson model is used for unearned exposure?

- IFRS17 – not used at all
- Solvency II – we can calculate expected claims using compound Poisson model, by just modifying T in the intensity if the Poisson distribution

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Earned exposure – process of claim handling and reserving

Damage

- Damage covered by insurance occurs

Reporting

- Insured reports the damage to the insurance company

Claim handling

- **Claim handler** assesses the claim
- If necessary **claim adjuster** (*takstmann*) is sent to evaluate the damage
- Claim handler sets a **case reserve** as an expected cost of the claim

Claim settlement

- The claim is being settled (e.g. building rebuilt after a fire or compensation is paid)
- The cost of the claim can deviate from the case reserve
- Case reserve changed/reduced as claim is being settled

Closing of the claim

- When claim is settled, the case is closed.
- Case reserve is set as 0.
- It can be of course reopened if the damage turns out to be settled insufficiently

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Reserve for earned exposure

Solvency II

(Best estimate of) Claim Provision =
discounted sum of expected claims and of expected expenses connected to earned exposure from all the policies, regardless when the policy was written

Solvency II has also an element of reserves called Risk Margin which serves the same purpose as risk adjustment, but is calculated for both unearned and earned exposure and with different method. However, we are not that interested here in risk margin/adjustment and we will focus just on Claim Provision

IFRS17

Liability for incurred claims =
discounted sum of expected claims and of expected expenses connected to earned exposure from all the policies, regardless when the policy was written

PLUS

risk adjustment which measures the uncertainty of those amounts claims

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From claim handling to reserve for earned exposure

- **Total reserve** (undiscounted) = **RBNS** + **IBNR + Expenses**
- **RBNS** (reported but not settled) = sum of all the case reserves for open claims – comes from claim handling division
- **IBNR** (incurred but not reported) = the reserve calculated by an actuary for all the claims that happened but have not been yet reported and for claims that have been reported but the real cost can be larger than the case reserve (i.e. reported insufficiently)
- **Expenses** come e.g. from the budget

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How to model IBNR

- Naïve approach – from a priori estimate:
  - Compound Poisson model says that we should expect X kr of claims, we have paid out Y kr and set Z kr as remaining case reserves/RBNS, therefore IBNR = X-Y-Z
  - Does not take into account the information in the case reserves or paid claims. Maybe be useful for exposure very recently earned, when the payment data or case reserves give little additional info,
- "Payment pattern approach" – from analysis how fast the claims are settled
  - at time t we usually pay out x% of final claim, so if we paid out Y so far, then IBNR = Y/x-Y-Z
  - May give excessively volatile reserves
  - Best for exposure where payment data or case reserves sufficiently reliable
- Combination of both
  - Weighting the naïve approach and payment pattern approach, giving naïve approach more weight for newly earned exposure and payment pattern approach for older exposure

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How to compute payment pattern? – outside the curriculum

Starting point – a claim triangle, which groups the payments attributed to claims in two dimensions: accident period (i.e. period when damage happened) and development period (i.e. period, measured from the accident date, when a particular payment was made)

E.g. gross claim triangle from If P&C Insurance from the regulatory reporting

(https://www.sampo.com/globalassets/arkisto/hallintomateriaali/sfcr/2021/quantitative-reporting-templates-2021-if-pc-insurance-ltd..xlsx Sheet S.19.01.21)

Development year

Gross Claims Paid (non-cumulative) - Development year (absolute amount). Total Non-Life Business

Accident year

		0	1	2	3	4	5	6	7	8	9	10 & +
		C0010	C0020	C0030	C0040	C0050	C0060	C0070	C0080	C0090	C0100	C0110
Prior	R0100											217 125
2012	R0160	14 474 654	7 033 205	1 634 993	545 931	413 564	271 824	171 518	115 952	79 579	52 941	
2013	R0170	14 421 745	6 441 371	1 284 522	766 898	457 537	366 395	210 043	110 824	139 251		
2014	R0180	14 481 434	5 948 468	1 248 430	925 694	453 607	297 128	323 544	119 717			
2015	R0190	14 486 213	6 051 431	1 436 470	661 912	411 034	257 075	181 484				
2016	R0200	14 948 147	6 437 069	1 341 877	705 255	392 564	365 743					
2017	R0210	15 477 478	6 574 683	1 747 644	801 530	296 406						
2018	R0220	17 204 085	7 933 821	1 618 621	639 792							
2019	R0230	18 546 108	7 277 424	1 497 789								
2020	R0240	17 487 361	7 067 644									
2021	R0250	17 508 967										

Gross Claims Paid (non-cumulative) - Current year, sum of years (cumulative). Total Non-Life Business

		In Current year	Sum of years (cumulative)
		C0170	C0180
Prior	R0100	217 125	217 125
2012	R0160	52 941	24 794 161
2013	R0170	139 251	24 198 586
2014	R0180	119 717	23 798 023
2015	R0190	181 484	23 485 618
2016	R0200	365 743	24 190 655
2017	R0210	296 406	24 897 741
2018	R0220	639 792	27 396 319
2019	R0230	1 497 789	27 321 321
2020	R0240	7 067 644	24 555 005
2021	R0250	17 508 967	17 508 967
Total	R0260	28 086 858	242 363 522

Diagonal = payments carried out in the same calendar year, but for claims from not necessarily the same accident year

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How to compute payment pattern? – Chain Ladder - outside the curriculum

Take a historical CUMULATIVE claim triangle

$$C = [C_{ij}]_{\substack{i-accident\ period \\ j-development\ period}} = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n-1} & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-1,1} & C_{n-1,2} & & 0 & 0 \\ C_{n,1} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

, where C_{ij}= cumulative amount of all payments for i-th accident period and until the j-th development period.

Define then j-th development factor $\hat{f}_j$ as

$$\hat{f}_j = \frac{\sum_{i=1}^{n-j} C_{i,j+1}}{\sum_{i=1}^{n-j} C_{i,j}}.$$

Measures how much on average the cumulative claim amounts increase from one development period to another

Then the best estimate of a payment for i-th accident period and j-th development period (assume i+j>n+1, i.e. we are in the lower part of the triangle) is:

$$\hat{P}_{i,j} = \underbrace{\mathrm{C}_{\mathrm{i,n-i+1}} * \hat{f}_{n-i+1} * \cdots * \hat{f}_{j-1}}_{\text{Cumulative claim amount till development period j}} - \underbrace{\mathrm{C}_{\mathrm{i,n-i+1}} * \hat{f}_{n-i+1} * \cdots * \hat{f}_{j-2}}_{\text{Cumulative claim amount till development period j-1}}$$

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Chain Ladder – example (done during exercise classes)

- See the Excel sheet for the example for If P&C Insurance

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Payment pattern for a priori distribution and discounting of premium provision– outside curriculum

- Premium provision is discounted sum of expected claims and of expected expenses minus discounted sum of expected future premiums connected to unearned exposure from all the active policies
- So far we only considered undiscounted total amount. How to discount it?
- We know when we are to get premiums and pay expenses – no problem with discounting
- Only problem with when claims are to be paid => use payment pattern from Chain Ladder
- To be more exact: easy to check that if $\hat{f}_j$ is j-th development factor, j=1,..,n, then the percentage paid in j-th year can be calculated as:

$$a_j = \frac{\prod_{i=1}^{j-2} \hat{f}_i(\hat{f}_{j-1}-1)}{\prod_{i=1}^{n-1} \hat{f}_i}$$

- Then we can just multiply the total amount of claims by a_j to get the estimate of the cashflow in year j-th. Having cashflows, we can discount them.

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Loss ratio and combined ratio

- Two very important ratios describing the performance and profitability of portfolio
- Loss ratio="incurred claims in the period" over "premiums earned in the period"
- $Loss\ ratio = \frac{claims\ paid\ out\ in\ the\ period + change\ in\ claim\ provision\ in\ the\ period}{earned\ premiums}$
- Combined ratio="incurred claims and expenses in the period" over "premiums earned in the period"
- $Combined\ ratio =$
  $\frac{claims\ paid\ out\ in\ the\ period\ +\ expenses\ in\ the\ period +\ change\ in\ claim\ provision\ in\ the\ period}{earned\ premiums}$
- Loss ratio in Scandinavia is ca. 70%. Combined ratio is ca. 85%
- If combined ratio is over 100% then the company is losing money on insurance activities

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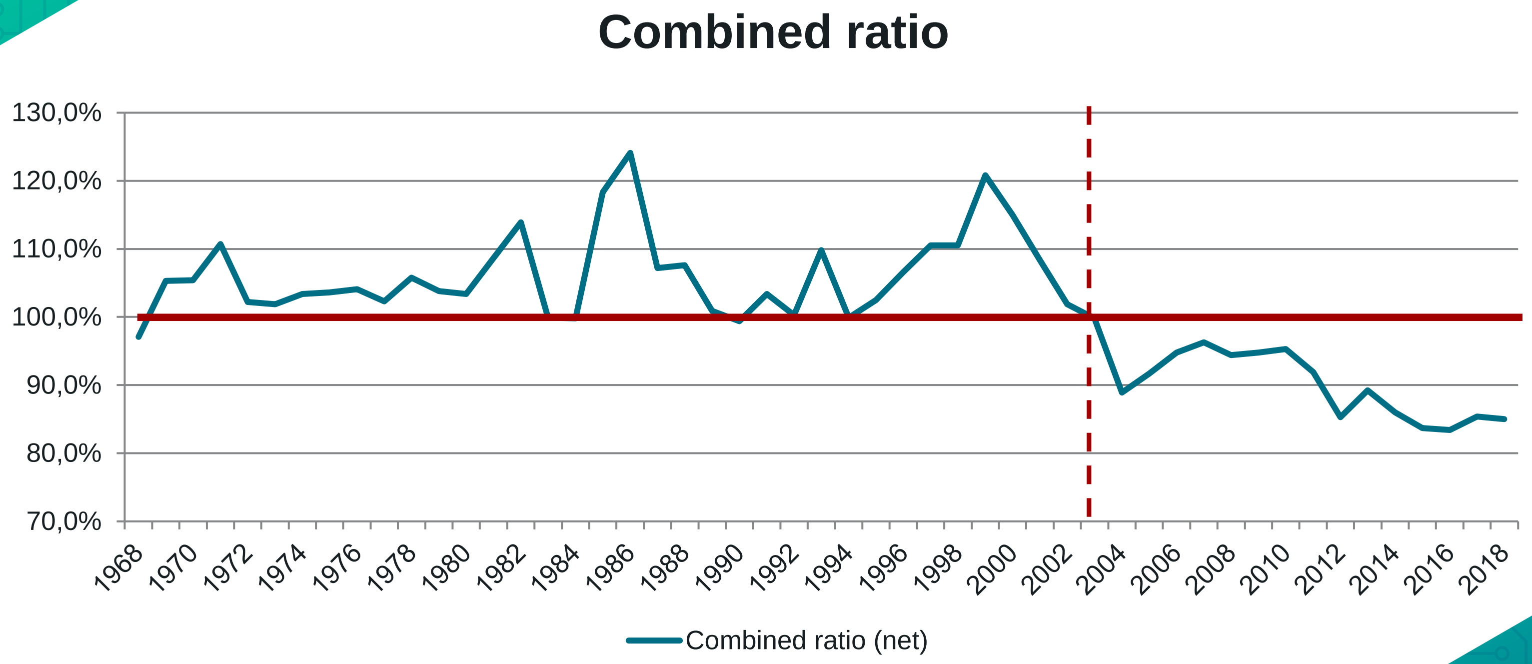
Historical development of combined ratio for Gjensidige Forsikring



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Pricing of an insurance contract in real life

Risk differentiation

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Pricing so far

- We assumed a compound Poisson model, either on portfolio basis or on a policy basis, and we calculated pure premium based on it.
- For model per policy, we assumed that intensity of the Poisson process and the parameters of the claim severity distribution may depend on policy
- In this section we wonder how we can come up with the parameters per policy

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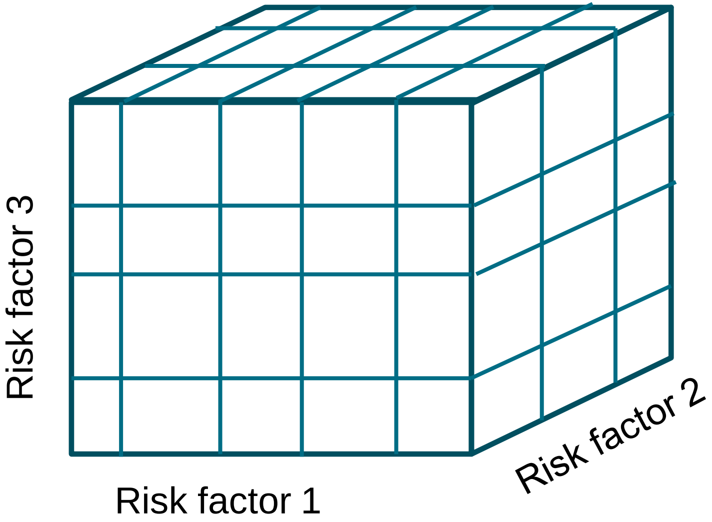
Risk differentiation

- Assume that we suspect that some variables can determine the frequency of claims, e.g. a young driver tends to drive faster than an older one and therefore causes more accidents. We call such a variable a **risk factor**
- **Risk differentiation** is a process to determine such risk factors and divide the portfolio into a small groups with similar risks. Those groups will get the same pure premium
- Risk differentiation can be done both for the claim frequency and claim size. We will focus mostly on modelling risk differentiation for claim frequency



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Motivation – exponential family of distributions (outside curriculum)

- Exponential family distribution contains the distributions (discrete and continuous) for which the probability function can written as:

$$f(y) = c(y, \phi) \exp\left\{\frac{y\theta - a(\theta)}{\phi}\right\}$$

- $\theta \in \mathbb{R}$ is called a canonical parameter telling something about the location, ϕ is a dispersion parameter, $\mathrm{a}(\theta)$ and $\mathrm{c}(\mathrm{y}, \phi)$ are function
- We know moreover that if X is distributed according to f then
  - $E(X) = \frac{d}{d\theta} a(\theta) \quad and \quad Var(X) = \phi \frac{d^2}{d\theta^2} a(\theta)$
- For Poisson we know that the probability function can be written as

$$p(k, \lambda) = \frac{\lambda^k}{k!} e^{-\lambda} = \frac{1}{k!} e^{k \ln \lambda} e^{-\lambda} = \frac{1}{k!} e^{k \ln \lambda - \lambda}$$

Comparing the terms we get that

$$c(y, \phi) = \frac{1}{y!}, \qquad \theta = \ln \lambda, \qquad a(\theta) = e^{\theta}, \qquad \phi = 1$$

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Motivation – generalized linear models (outside the curriculum)

- We assume that our frequency depends on some risk factors $X_1, \dots, X_n$
- E.g. for car insurance $X_1, \dots, X_n$ can be an age of the driver, the power of the engine etc.
- In regular regression we want to model:

$$E(Y|X_1, \dots, X_n) = \alpha_0 + \alpha_1 X_1 + \dots + \alpha_n X_n$$

- However, regular regression can produce negative numbers, therefore its not a good idea if Y is Poisson
- However, we know that for Poisson

$$E(Y|X_1, \dots, X_n) = \frac{d}{d\theta} a\left(\theta_{|X_1,\dots,X_n}\right) = \exp \theta_{|X_1,\dots,X_n}$$

And θ can take all real values and can be modelled therefore with a linear regressors

Therefore we define our model as

$$E(Y|X_1, \dots, X_n) = \exp \theta_{|X_1,\dots,X_n} = \exp(\alpha_0 + \alpha_1 X_1 + \dots + \alpha_n X_n)$$

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Poisson regression. Why it is nice?

- Assume the model $E(N|X_1, \dots, X_n) = \exp(\alpha_0 + \alpha_1 X_1 + \cdots + \alpha_n X_n), N \sim Poiss(\lambda)$
- $E(N|X_1, \dots, X_n) = \exp(\alpha_0 + \alpha_1 X_1 + \cdots + \alpha_n X_n) = \exp(\alpha_0)\exp(\alpha_1 X_1) * \cdots * \exp(\alpha_n X_n)$
- Then $\exp(\alpha_0)$ is base frequency, while $\exp(\alpha_i X_i)$ are loadings saying how much the real frequency will fall/increase compared to the base frequency for a particular value of the risk factor X_i
- Very easy to interpret and to check the reasonableness of the model
- Typical variables :
  - For the car insurance: age of driver, the power of engine, age of car, experience of the driver, number of claims in the past, address of the driver, value of the car…
  - For property insurance: age of a building, number of storeys, areal, construction type, type of roof, number of storeys underground, number of people in the household, location, number of claims in the past, is it rented out or not….

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From regression to pure premium

- Similar regression can be made for average claim severity (note that depending on the claim distribution we may have different “link” between regressors $X_1, \dots, X_n$ and $E(Z_1 | X_1, \dots, X_n)$
- Then we use the result that

$$\pi^{pu} = EX = \mu_i T * E(Z_1) = T\, E(N | X_1, \dots, X_n)\, E(Z_1 | X_1, \dots, X_n)$$

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Compound Poisson with unknown intensity

Classified: General Business

Book: chapter 6, section 6.2.5 and 6.3.5

Compound Poisson model revisited

So far we assumed that $\text{Claims} = X = \sum_{i=1}^{N} Z_i$, where N random variable with values 0,1,2,... and Z_i $i.i.d.$, independent of N. Moreover usually we assumed that $N \sim Poiss(\lambda)$ and we assumed that we know λ.

What happens if λ is unknown? We get then a model (compound Poisson with intensity uncertainty):

1. $X = \sum_{i=1}^{N} Z_i$,
2. Z_i $i.i.d.$,
3. $N \sim Poiss(\lambda)$, where λ is a random variable independent of Z_i's and N
4. N is independent of Z_i's

Moments of such model – via conditional expectation

By rule of double expectation:

$$E(X) = E\big(E(X|\lambda)\big) = E(\lambda E(Z_i)) = E(\lambda)\,E(Z_i)$$

By rule of double variance:

$$Var(X) = E\big(Var(X|\lambda)\big) + Var\big(E(X|\lambda)\big) =$$
$$= E[\lambda(Var(Z_i)+(EZ_i)^2)\,] + \mathrm{Var}\big(\lambda E(Z_i)\big) =$$
$$= (Var(Z_i)+(EZ_i)^2)\mathrm{E}(\lambda) + Var(\lambda)(EZ_i)^2$$

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Lets look at the ratio sd/E

Assume now that $\lambda = J\mu T$, where μ is a random variable, independent of N and Z_i's

If we look at the ratio of standard deviation over expectation, we get

$$\frac{sd(X)}{E(X)} = \frac{\sqrt{JT(Var(Z_i)+(EZ_i)^2)\,E(\mu) + J^2\,T^2\,Var(\mu)(EZ_i)^2}}{JTE(\mu)\,E(Z_i)} =$$

$$= \sqrt{\frac{JT(Var(Z_i)+(EZ_i)^2)\,E(\mu)}{J^2T^2(E(\mu)E(Z_i))^2} + \frac{J^2\,T^2\,Var(\mu)(EZ_i)^2}{J^2T^2(E(\mu)E(Z_i))^2}} =$$

$$\sqrt{\frac{(Var(Z_i)+(EZ_i)^2)}{JTE(\mu)(E(Z_i))^2} + \frac{Var(\mu)}{(E(\mu))^2}}$$

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Conclusion

For Compound Poisson model with uncertainty regarding the intensity of the Poisson variable, the risk per unit of exposure can be decomposed into

- a diversifiable risk which can be reduced to zero by increasing the number of policies in the portfolio
- and undiversifiable risk, which cannot be diversified away and is connected to the uncertainty of the intensity parameter.

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Life insurance

Two state (alive/dead) model

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Book: chapter 3 section 3.4

Chapter 6, section 6.2.3

Life insurance

- Insurance which protects against events over a long term horizon where the uncertainty is both **whether** and **when** the trigger event occurs
- Example of the trigger:
  - dying,
  - surviving till a particular age,
  - getting widowed,
  - getting disabled.
- What type of long term horizon? Many years, in many cases till death.

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Risk sharing for life insurance

- For non-life insurance risk sharing was relying on customers paying premium against some risks which might happen with low probability but if they occur the amount is excessive for an individual to bear on their own. The premiums from customers without claims are used to settle the damage of those unlucky ones
- In life insurance, risk share is relying on customer paying premiums over a longer time period against risks for which both occurrence and timing is uncertain. Life insurance "transfers" premiums both between customers (e.g. "surviving customers" covering the benefit for those who died via premiums) and between periods (e.g. customers saving money for retirement)

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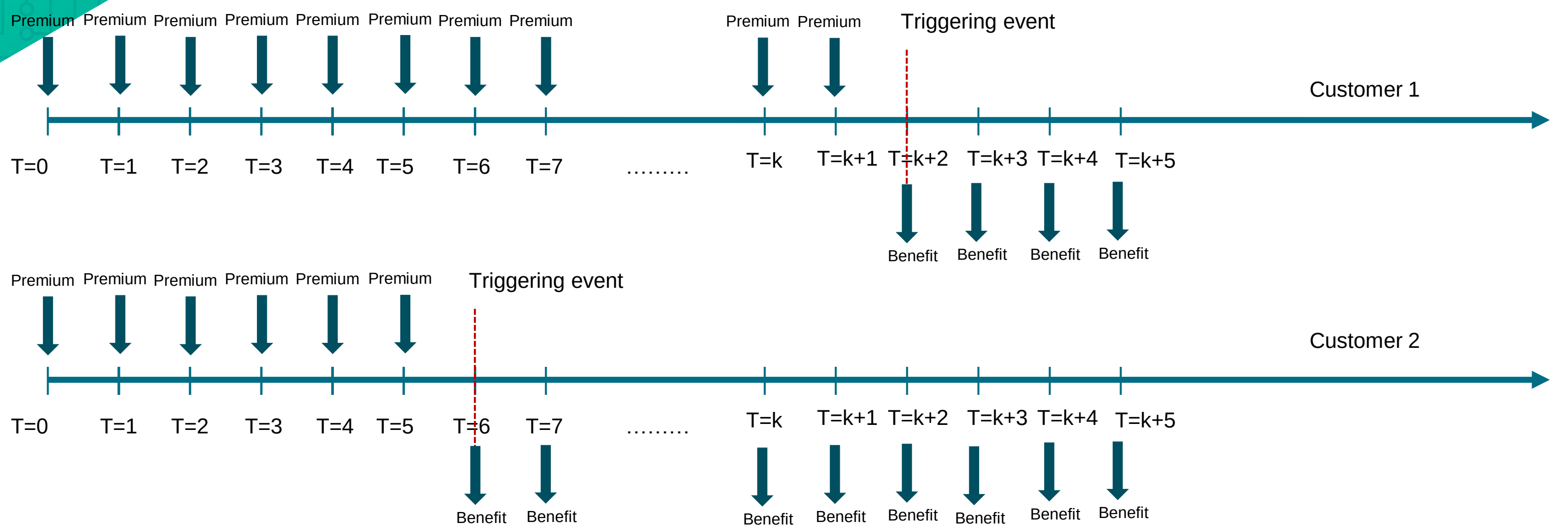
Risk sharing in practice



For customer 2 triggering event occurs much faster and therefore she pays in much less premiums than customer 1 and gets much more in benefits.

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Modelling in life insurance – main idea

- We deal with cashflows over a long time period => we need **discounting**!
- Cashflow amounts are uncertain as we don't know when the triggering event happens => we need to deal with **uncertainty** somehow.
- We have risk sharing, which means that if portfolio is very large, the total paid benefit amount from the portfolio is going to be equal expected value of paid benefits due to LLN.
- So the main modelling tool for life insurance is **expected present value** of the cashflows

- $V_0 = PV = \sum_{i=0}^{\infty} d^i E(X_i)\,, \;\; where$
- $d^i - discount\ factor\ for\ period\ i\ \left(e.g.\, d^i = P_0(0:i)\ or\ d^i = \frac{1}{(1+r)^i}\right),$
- $E(X_i) - expected\ value\ of\ the\ cashflow\ at\ time\ i$

Classified: General Business

Simple examples: annuity/defined benefit pension

- Defined benefit pension – pays benefit s every year if an insured survives till a retirement age
  - l_0- current age of insured
  - l_r -retirement age
  - ${}_kp_{l_0}$-probability of surviving k-next periods if your current age is l_0
  - π-premium paid every year until retirement (or death)
- Note that if you are dead there is no payment (either of premiums or pensions)
- Therefore expected cashflow is of the form ${}_kp_{l_0} * x + \left(1 - {}_kp_{l_0}\right) * 0 = {}_kp_{l_0} * x$
- Then:
- $V_0 = \sum_{i=0}^{l_r-l_0-1} d^i * (-\pi) * {}_ip_{l_0} + \sum_{i=l_r-l_0}^{\infty} d^i * s * {}_ip_{l_0}$

$$V_0 = \underbrace{\sum_{i=0}^{l_r-l_0-1} d^i * \underbrace{(-\pi) * {}_ip_{l_0}}_{\text{Expected payment during contribution stage}}}_{\text{Contribution stage}} + \underbrace{\sum_{i=l_r-l_0}^{\infty} d^i * \underbrace{s * {}_ip_{l_0}}_{\text{Expected payment during receiving stage}}}_{\text{Receiving stage}}$$

Classified: General Business

Lets formalize the model a bit

- Assume that the insured can be in two states $S = \{*, \dagger\}$. E.g.
  - State * - the insured is alive
  - State † - the insured is dead
- For each i=0,…,k,… let X_i denote a random variable describing the state of the insured after i years, and therefore
  $X_i \in S = \{*, \dagger\}$. Sequence $X = (X_i)_{i=0,1,\dots}$ is a stochastic process.
- If $S = \{*, \dagger\}$ and * - the insured is alive, † - the insured is dead then $P(X_{i-1} = \dagger, X_i = *) = 0$. The process X is therefore going to consist of trajectories $(*,*,*,\dots,*,\dagger,\dagger\,\dagger,\dots)$.

Some remarks:

- We will later extend the model to take into account more general situation, e.g. when one of the states can be disability
- The state space S can also be even more complex: let's assume e.g. assume an "orphan" insurance when the a child gets a benefit whenever both parents die. Then we need to model $S = \{(*,*), (\dagger,*), (*,\dagger), (\dagger,\dagger)\}$

Classified: General Business

Policy functions

- The payments which both sides need to make depend on the state of the process X
- Policy functions define the cashflows assumed in the policy given different state of X. Both cashflows in (i.e. premiums) and cashflows out (i.e. benefits) are taken into account.
- Two types of payments:
  - **Payments due to being at the state s.** E.g. pension: if you are alive and under an agreed retirement age, you need to pay premium, but if you are alive and over the agreed age you get a pension from insurance company
  - **Payments due to transitioning from state r to state s.** E.g. if you have a death insurance, if you die (i.e. transition from alive to dead) and are under a pre-agreed age, insurance company is going to pay a death benefit to your family.
- Therefore we need to define two policy functions $a_s(i), s \in S$ and $a_{r,s}(i), s \neq r \in S$. The policy function defines a cashflow at time i given that $X_i = \text{s}$, the other one defines a cashflow at time I given that X transitioned from state r to s, i.e. $X_{i-1} = \text{r}, X_i = \text{s}$.
- Then $V_0 = PV = \sum_{i=0}^{\infty} d^i \left(\sum_{s \in S} (a_s(i) P(X_i = s) + \sum_{\substack{r \in S \\ r \neq s}} a_{r,s}(i) P(X_{i-1} = r, X_i = s)) \right)$

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Policy functions for different contracts – term insurance

- Term insurance pays a one-time given benefit B if an insured, current age l_0, dies before a pre-agreed age l_d.
- Policy functions have a form
  - $a_*(i) = \begin{cases} -\pi & if\ i \leq l_d - l_0 - 1 \\ 0 & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_{*\dagger}(i) = \begin{cases} B & if\ i \leq l_d - l_0 \\ 0 & otherwise \end{cases}$

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Policy functions for different contracts – pure endowment

- Pure endowment insurance pays a one-time given benefit B if an insured, current age l_0, survives until a pre-agreed age l_r. It can be seen as a pension paid in advance
- Policy functions have a form
  - $a_*(i) = \begin{cases} -\pi & if\ i \le l_r - l_0 - 1 \\ B & if\ i = l_r - l_0 \\ 0 & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_{*\dagger}(i) = 0$

Policy functions for different contracts –endowment

- Endowment insurance pays a one-time given benefit B_s if an insured, current age l_0, survives until a pre-agreed age l_r, or one-time given benefit B_d if the insured dies before a pre-agreed age l_r . It can be seen as a pension paid in advance with death insurance.
- Policy functions have a form
  - $a_*(i) = \begin{cases} -\pi & if\ i \leq l_r - l_0 - 1 \\ B_s & if\ i = l_r - l_0 \\ 0 & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_{*\dagger}(i) = \begin{cases} B_d & if\ i \leq l_r - l_0 \\ 0 & otherwise \end{cases}$

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Policy functions for different contracts –defined benefit pension

- Defined benefit pays a lifelong benefit s if an insured, current age l_0, survives until a pre-agreed age l_r. Policy functions have a form
  - $a_*(i) = \begin{cases} -\pi & if\ i \leq l_r - l_0 - 1 \\ s & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_{*\dagger}(i) = 0$

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Modelling of X

- In order to model X we need to investigate probability of survival
- Let Y denote how long an individual lives.
- Let ${}_kp_{l_0}$-probability of surviving (at least) k-next periods if your current age is l_0.
- Then ${}_kp_{l_0} = P(Y > l_0 + k|Y > l_0)$
- Define p_i - probability that an individual survives (at least) one year given that her current age is i. Define q_i- probability that an individual dies within next year given her current age is i.
- Therefore p_i= ${}_1p_i$ and q_i = $1 - p_i$.

- Nota bene: life expentancy = E(Y)

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Modelling of X

- We can moreover write ${}_kp_{l_0}$ in terms of p_i
- $${}_kp_{l_0} = P(Y > l_0 + k|Y > l_0) = \frac{P(Y>l_0+k)}{P(Y>l_0)} = \frac{P(Y>l_0+k)}{P(Y>l_0+1)}\frac{P(Y>l_0+1)}{P(Y>l_0)} =$$
$$\frac{P(Y>l_0+k)}{P(Y>l_0+k-1)}\frac{P(Y>l_0+k-1)}{P(Y>l_0-k-1)}\dots\frac{P(Y>l_0+2)}{P(Y>l_0+1)}\frac{P(Y>l_0+1)}{P(Y>l_0)} = p_{l_0}p_{l_0+1}\dots p_{l_0+k-1}$$

- Note that it is also easy to calculate the probability of dying after k years
- $$P(Y = l_0 + k|Y > l_0) = \frac{P(Y=l_0+k)}{P(Y>l_0)} = \frac{P(Y=l_0+k)}{P(Y>l_0+1)}\frac{P(Y>l_0+1)}{P(Y>l_0)} =$$
$$\frac{P(Y=l_0+k)}{P(Y>l_0+k-1)}\frac{P(Y>l_0+k-1)}{P(Y>l_0-k-1)}\dots\frac{P(Y>l_0+2)}{P(Y>l_0+1)}\frac{P(Y>l_0+1)}{P(Y>l_0)} = p_{l_0}p_{l_0+1}\dots p_{l_0+\,k-2}(1-p_{l_0+k-1})$$

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Modelling of X

- Connection with our process X:
- $X = (X_i)_{i=0,1,\dots}$ where X_i defines a state of an insured after i years.
- Assume the age of the insured is l_0 at the beginning of the contract. Then
  - $P(X_i = *) = {}_ip_{l_0} = p_{l_0}p_{l_0+1} \dots p_{l_0+i-1}$
  - $P(X_i = \dagger) = 1 - P(X_i = *) = 1 - {}_ip_{l_0}$
  - $P(X_{i-1} = *, X_i = \dagger) = p_{l_0}p_{l_0+1} \dots p_{l_0+i-2}(1 - p_{l_0+i-1})$
- Now the only thing which we need to model is p_i's (or q_i's)
- Mortality tables – a table which containts q_i's.
- https://www.ssb.no/statbank/table/07902/

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More about the cashflows

- When discounting, important to know if the payment is paid at the beginning a period k (then discounted with d^k) or at the end of the period (then discounted with d^{k+1}).
- If the payment is paid at the beginning of the period, we say it is paid **in advance**. If the payment is paid at the end of the period, we say it is paid **in arrears**.
- Defined benefit pension paid in advance:

$$V_0 = PV = \sum_{i=0}^{l_r - l_0 - 1} d^i * (-\pi) * {}_i p_{l_0} + \sum_{i=l_r - l_0}^{\infty} d^i * s * {}_i p_{l_0}$$

- Defined benefit pension paid in arrears:

$$V_0 = PV = \sum_{i=1}^{l_r - l_0} d^i * (-\pi) * {}_i p_{l_0} + \sum_{i=l_r - l_0 + 1}^{\infty} d^i * s * {}_i p_{l_0}$$

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Pricing of the life-contract

- Usually life contracts have a periodic (monthly/yearly premiums). It is because it is a form of saving with risk sharing.
- For the sake of simplicity we assume only yearly premiums paid in advance or arrears
- In non-life insurance we had a notion of pure premium, which was a premium which was covering the expected claims from a policy.
- Similar concept for life insurance: we want to have a **fair premium**, i.e. premium for which $V_0 = 0$. This premium reflects the expected level of payment given the estimates of mortality.
- The principle that premiums give $V_0 = 0$ is called **equivalence principle.**

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Example – defined benefit pension paid in advance

$$V_0 = \sum_{i=0}^{l_r-l_0-1} d^i * (-\pi) * {}_ip_{l_0} + \sum_{i=l_r-l_0}^{\infty} d^i * s * {}_ip_{l_0} = 0$$

$$\pi \sum_{i=0}^{l_r-l_0-1} d^i * {}_ip_{l_0} = s \sum_{i=l_r-l_0}^{\infty} d^i * {}_ip_{l_0}$$

$$\pi = s \frac{\sum_{i=l_r-l_0}^{\infty} d^i * {}_ip_{l_0}}{\sum_{i=0}^{l_r-l_0-1} d^i * {}_ip_{l_0}}$$

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Reserving of the life-contract

- One could also have another principle for premiums: **one-time premium paid in advance.**
- Then using the equivalence principle one could come up with this one-time premium.
- This quantity has a very interesting interpretation – as a **mathematical reserve**: it tell us how much money we need to set aside as a reserve for the policy
- Interestingly, a similar definition is used when setting reserves for accounting. Note that the is meaning in distinguishing between earned and unearned exposure as most of it unearned.
- For solvency purposes one uses a different definition: discounted expected value of benefits (and expenses) minus discounted expected value of future premiums – Very similar to definition of premium provision in non-life insurance, but completely different meaning!!!

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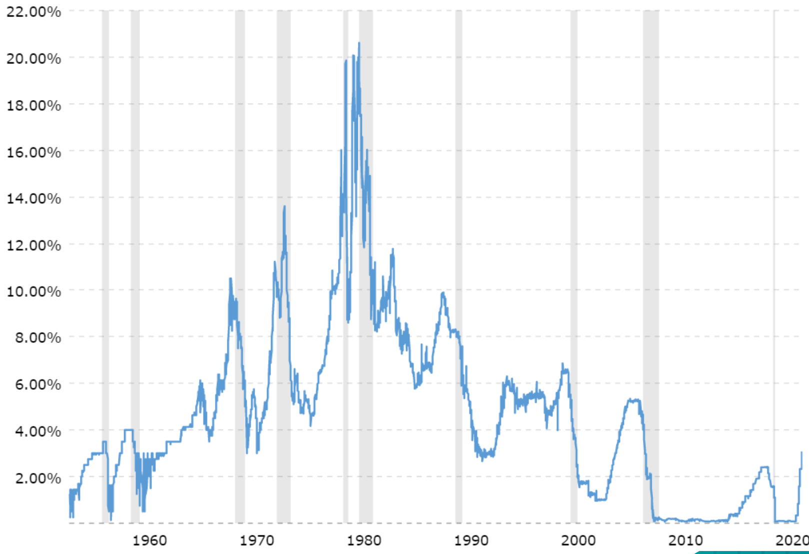
About discounting

- Historically one used a fixed “technical interest rate” to discount cashflows in life insurance.
- Technical interest rate was “prudent”, i.e. lower than the market rate. One thought that going under it was rather improbable.
- Advantage of technical interest rate – no volatility in reserves due to changes in the interest rate
- Disadvantage: if interest rate falls under the technical rate one may hold in real life too small reserve to cover future benefits
- However, the interest rate went down and down to ultralow level in 2010’s. Life insurance companies in trouble and need to change the technical interest rate

US central bank interest rate (Federal Fund rate)



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Fair value calculation and consequences

- Due to that experience more and more often to use market interest rates to discount, i.e. discounting e.g. with the zero-coupon interest rates.
- Advantage: reflects the current level of interest rate so no hidden losses
- Disadvantage: interest rate changes from day to day. The life insurance cashflows have very long duration so they are super-sensitive to changes in discounting rates.
- Capital requirement (i.e. capital which insurance companies need to hold to cover losses) much higher due to that (and other risks, e.g. longevity) . Insurance companies move to less capital-intensive products such as saving/investment.

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Modelling of mortality

Life tables and mortality laws

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Life/mortality tables

- Statistical tables – mortality in a predefined calendar year for different ages.
  - Not the best estimator for future mortality as does not take into account trends in mortality

- Life/mortality table for an individual – takes can take into account future trend in longevity
  - Usually based on some kind of "mortality law", which smooths the random perturbances in the data and may take into account trends

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Modelling of mortality again – mortality law (in continuous time)

- Let Y be a length of life of an individual and assume Y is taking values in a positive line
- Consider a probability $P(Y \leq x + h | Y > x)$ – probability of survival of h additional time.
- Define the mortality intensity as $\mu(x) := \lim_{h \to 0^+} \frac{P(Y \leq x+h | Y > x)}{h}$.
- Interpretation – probability of dying during the very short time period per unit of time, i.e. rate of dying
- One can prove then that ${}_t p_{l_0} = \exp(-\int_{l_0}^{l_0+t} \mu(s) ds)$

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Proof

$$\frac{d}{dx^+}P(Y \le x) = \lim_{h\to 0^+}\frac{P(Y \le x+h) - P(Y \le x)}{h} = \lim_{h\to 0^+}\frac{P(x < Y \le x+h)}{h}$$

$$= \lim_{h\to 0^+}\frac{P(Y > x, Y \le x+h)}{h} = P(Y > x)\lim_{h\to 0^+}\frac{P(Y \le x+h|Y > x)}{h} = P(Y > x)\mu(x)$$

Therefore we have

$$\frac{d}{dx^+}P(Y > x) = \frac{d}{dx^+}\big(1 - P(Y \le x)\big) = -P(Y > x)\mu(x)$$

Let $S(x) = P(Y > x)$.

Then we have ODE

$$\frac{d}{dx^+}S(x) = -S(x)\mu(x), S(0) = 1$$

Note that $S(x) = \exp(-\int_0^x \mu(s)ds)$ is the solution. Now note that ${}_tp_{l_0} = S(l_0 + t)/S(l_0)$ to get the result.

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Gomperz-Makeham mortality law in continuous time

- Assume that $\mu(x) = \theta_0 + \theta_1 e^{\theta_2 x}$. This is the intensity of Gomperz-Makeham probability law
- Easy to prove that ${}_tp_{l_0} = \exp\left(-\theta_0 t - \frac{\theta_1}{\theta_2}\left(e^{\theta_2(l_0+t)} - e^{\theta_2 l_0}\right)\right)$
- The difference between this law and the exercises: continuous vs. discrete time.

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Mortality and life time improvement

- Note that Gomperz-Makeham does not depend on the “time” in the sense that a 60-year old today and a 60-year old in 20 years will have the same mortality.
- Not realistic. One need to adjust to take care of that
- E.g. K2013 – real mortality table which is required by Finanstilsynet to be used in some reserving

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K2013 – not included curriculum

Dødeligheten for både opplevelsesrisiko og dødsrisiko for medlemmer i forsikret bestand er gitt ved følgende formler:

$$\mu_{Kol}(x,t) = \mu_{Kol}(x,2013) * (1 + \frac{w(x)}{100})^{t-2013}$$

Her er $\mu_{Kol}(x,2013)$ dødeligheten for et medlem i alder x i 2013, mens $\mu_{Kol}(x,t)$ er dødeligheten for et medlem i alder x i kalenderår t (for t minst lik 2013). Videre benevnes dødelighetsnedgangen med $w(x)$, der

$$w(x) = \min(2{,}671548 - 0{,}172480 * x + 0{,}001485 * x^2, 0) \quad \text{for menn}$$
$$w(x) = \min(1{,}287968 - 0{,}101090 * x + 0{,}000814 * x^2, 0) \quad \text{for kvinner}$$

Dødeligheten for opplevelsesrisiko er gitt ved følgende formler:

$$1000 * \mu_{Kol}(x,2013) = (0{,}189948 + 0{,}003564 * 10^{0{,}051*x}) \quad \text{for menn}$$
$$1000 * \mu_{Kol}(x,2013) = (0{,}067109 + 0{,}002446 * 10^{0{,}051*x}) \quad \text{for kvinner}$$

Dødeligheten for dødsrisiko er gitt ved følgende formler:

$$1000 * \mu_{Kol}(x,2013) = (0{,}241752 + 0{,}004536 * 10^{0{,}051*x}) \quad \text{for menn}$$
$$1000 * \mu_{Kol}(x,2013) = (0{,}085411 + 0{,}003114 * 10^{0{,}051*x}) \quad \text{for kvinner}$$

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Multistate model

Modelling disability

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Book: chapter 6, section 6.5.1 - 6.5.3 and 6.6

Probabilistic model behind multistate model

A stochastic process $X = (X_i)_{i=0,1,\dots}$ is called a **Markov chain** with state space S if

1. $X_i \in S \; for \; all \; i$

2. $P(X_n = s_n | X_1 = s_1, X_2 = s_2, \dots, X_{n-1} = s_{n-1}) = P(X_n = s_n | X_{n-1} = s_{n-1}) =: \mathrm{p}_{s_{n-1}, s_n}(\mathrm{n})$, for any $s_1, \dots, s_n \in S$

The next state of Markov chain depends therefore only on the previous state, not on the whole history (Markov property).

Example of Markov process: discreet random walk

Let $S = \mathbb{Z}, \mathrm{X}_0 = 0, \mathrm{X}_\mathrm{i} = \mathrm{X}_{\mathrm{i}-1} + \mathrm{U}_\mathrm{i}$, where $U_i = \pm 1 \; iid, P(U_i = 1) = p$

Easy to see that

$$\mathrm{p}_{s,t}(n) = \begin{cases} p & if \; t = s+1 \\ 1-p & if \; t = s-1 \\ 0 & otherwise \end{cases}$$

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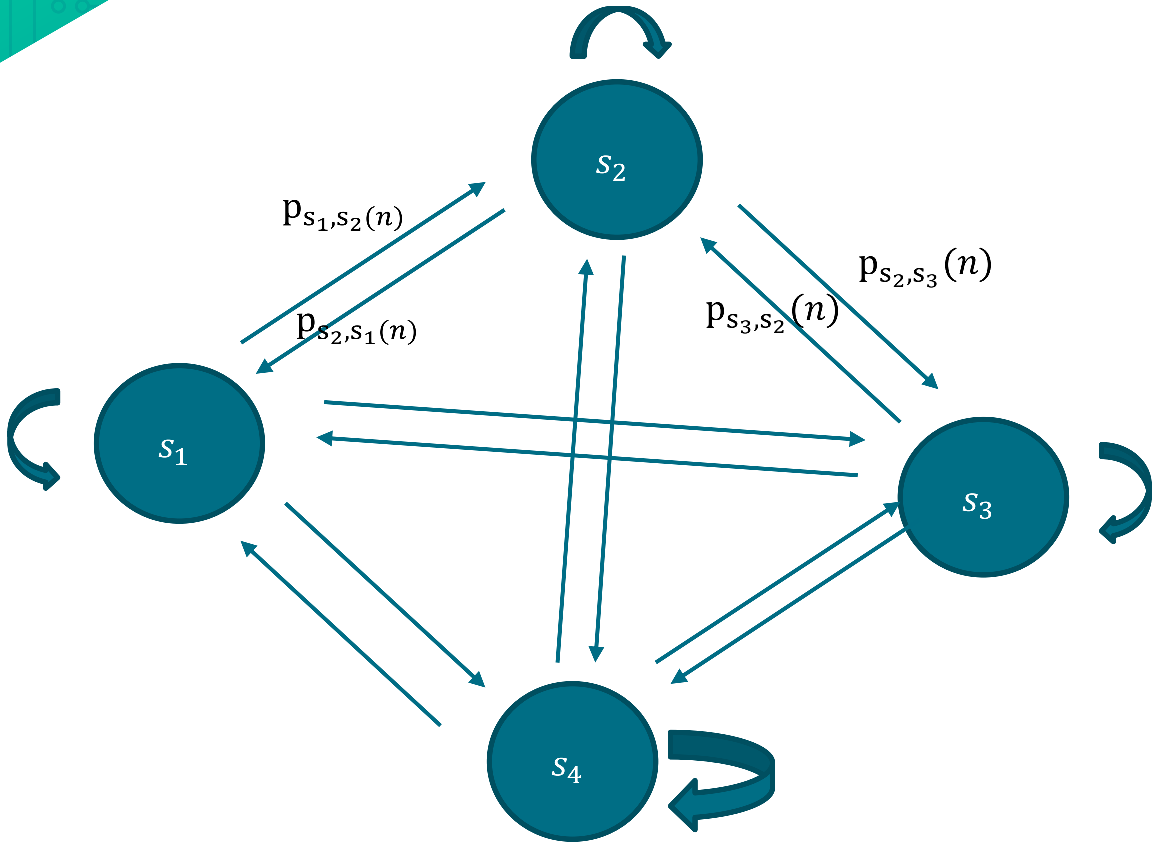
Graph of a Markov chain



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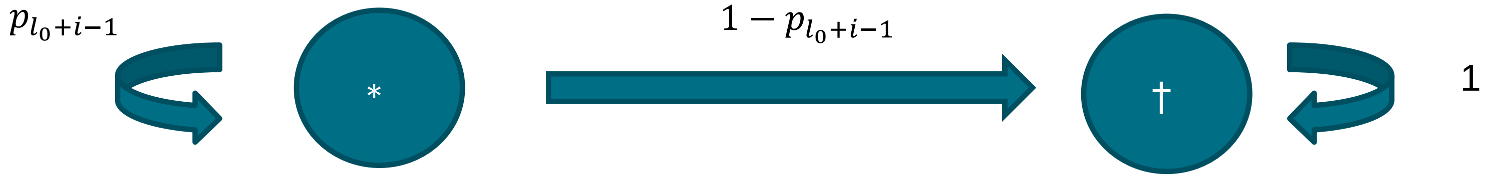
Two state model as a Markov chain

- State space $S = \{*, \dagger\}$.
- $P(X_i = \dagger | X_{i-1} = *) = (1 - p_{l_0+i-1})$
- $P(X_i = * | X_{i-1} = *) = p_{l_0+i-1}$
- $P(X_i = \dagger | X_{i-1} = \dagger) = 1$
- $P(X_i = * | X_{i-1} = \dagger) = 0$



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Extend a model from previous lecture

- Assume that the insured can be in three states $S = \{*, \dagger, \diamond\}$.
  - State * - the insured is alive and active
  - State ⋄ - the insured is alive but disabled
  - State † - the insured is dead
- For each i=0,…,k,… let X_i denote a random variable describing the state of the insured after i years, and therefore
  $X_i \in S = \{*, \dagger, \diamond\}$. Sequence $X = (X_i)_{i=0,1,\ldots}$ is a stochastic process.

We will need to have now the following policy functions

$$a_*(t), a_\dagger(t), a_\diamond(t) \;\; - policy\ functions\ for\ being\ in\ a\ state$$

$$a_{*\diamond}(t), a_{*\dagger}(t), a_{\diamond *}(t), a_{\diamond\dagger}(t) - policy\ functions\ for\ transitioning\ from\ one\ state\ to\ another$$

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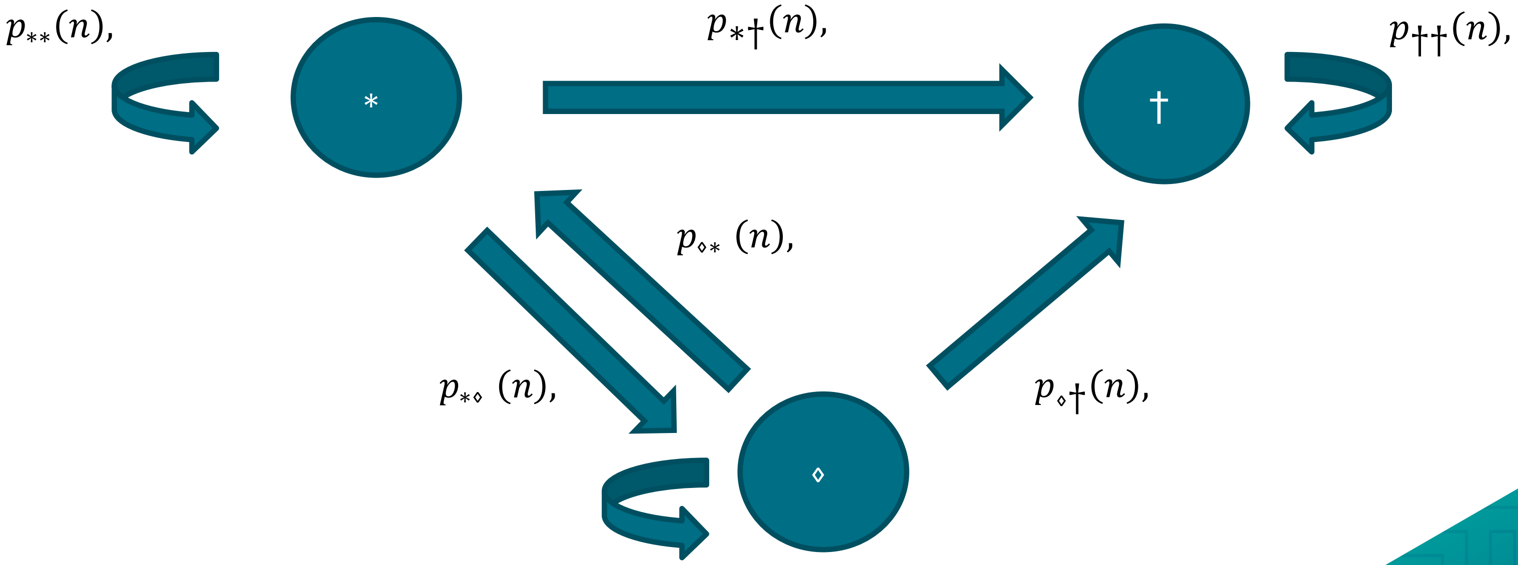
Markov chain behind a model



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How to define transition probabilities?

- We can assume independence between an event of surviving and getting disabled. Similarly we can assume independence between surviving and recovery from disability
- Then if $p_l(i|a)$ is probability of getting disabled at age l and $p_l(a|i)$ is probability of recovery from disability at time n then
- $P(X_n = \dagger | X_{n-1} = *) = (1 - p_{l_0+n-1})$
- $P(X_n = * | X_{n-1} = *) = p_{l_0+n-1}(1 - p_{l_0+n-1}(i|a))$
- $P(X_n = \diamond | X_{n-1} = *) = p_{l_0+n-1} p_{l_0+n-1}(i|a)$
- $P(X_n = * | X_{n-1} = \diamond) = p_{l_0+n-1} p_{l_0+n-1}(a|i)$
- $P(X_n = \diamond | X_{n-1} = \diamond) = p_{l_0+n-1}(1 - p_{l_0+n-1}(a|i))$
- $P(X_n = \dagger | X_{n-1} = \diamond) = (1 - p_{l_0+n-1})$
- $P(X_n = \dagger | X_{n-1} = \dagger) = 1$
- $P(X_n = * | X_{n-1} = \dagger) = 0$
- $P(X_n = \diamond | X_{n-1} = \dagger) = 0$

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Transition probability matrix

Note that we can put the probabilities into a matrix 3x3 $P(l_0+n-1)$. The rows would represent the state before transition, whereas the columns would represent the state after transition

$$P(l_0+n-1) = \begin{matrix} * \\ \diamond \\ \dagger \end{matrix} \overset{\begin{matrix} * & \diamond & \dagger \end{matrix}}{\begin{bmatrix} p_{l_0+n-1}(1-p_{l_0+n-1}(i|a)) & p_{l_0+n-1}p_{l_0+n-1}(i|a) & 1-p_{l_0+n-1} \\ p_{l_0+n-1}p_{l_0+n-1}(a|i) & p_{l_0+n-1}(1-p_{l_0+n-1}(a|i)) & (1-p_{l_0+n-1}) \\ 0 & 0 & 1 \end{bmatrix}}$$

Note that the sum in each row is 1

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An algorithm how to simulate Markov Chain

1. INPUTS:
   a. l_0 #start age
   b. n – number of states in the state space
   c. s_0 #starting state
   d. l_max # maximumum age
   e. P # a list with l_max-l_0-1 elements, element P[[i]] contains a transition probability matrix P(l_0+i-1)
   f. m # number of simulations
2. Set X = empty matrix s_0 with m rows and and length l_max-l_0 columns+1 #we will store the number of the simulated state in this matrix
3. Set X[,1]=s_0 #set a starting state equal for all simulations
4. For i in (1 to m) do
   a. For j in (1 to l_max-l_0)
      I. Draw U from uniform distribution between 0 and 1
      II. s=0,p=0
      III. Repeat
         I. s=s+1
         II. p=p + P[[j]] [X[i,j],s] # here we are operating on the appropriate row of the transition # matrix to find a state after transition

         Until p>U
      IV. X[i,j+1]=s
5. OUTPUT: return X

An algorithm how to calculate expected present value

1. INPUTS:
   a. l_0 #start age
   b. n – number of states in the state space
   c. s_0 #starting state
   d. l_max # maximumum age
   e. X # a Markov chain simulated before
   f. r # interest rate
   g. a_s # policy function from being in a state
   h. a_rs #policy function from transitioning
   i. m # number of simulations
2. Set PV =0, Y – empty vector of length m
3. For j in (0 to l_max-l_0) do
   a. Y=0
   b. For i in (1 to m) do
      I. Y[i]=a_s (l_0+j , X[i,j+1])
      II. If (X[i,j+1] <> X[i,j]) then Y[i]=Y[i]+a_rs (l_0+j , X[i,j] , X[i,j+1])
   c. PV=PV+1/(1+r)^j * mean(Y)
4. OUTPUT: return PV

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Examples of the policies - disability pension

- Disability pension pays a recurring benefit B from a period when an insured, current age l_0, becomes disabled and until a pre-agreed age l_r. Policy functions have a form:
  - $a_*(i) = \begin{cases} -\pi & if\ i \le l_r - l_0 - 1 \\ 0 & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_\diamond(i) = \begin{cases} B & if\ i \le l_r - l_0 \\ 0 & otherwise \end{cases}$
  - $a_{*\dagger}(i) = 0$
  - $a_{*\diamond}(i) = 0$
  - $a_{\diamond *}(i) = 0$
  - $a_{\diamond\dagger}(i) = 0$

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Examples of the policies - disability pure endowment

- Disability pure endowment pays a one time benefit B when an insured, current age l_0, gets disabled before a preagreed age l_r. Policy functions have a form:
  - $a_*(i) = \begin{cases} -\pi & if\ i \leq l_r - l_0 - 1 \\ 0 & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_\diamond(i) = 0$
  - $a_{*\dagger}(i) = 0$
  - $a_{*\diamond}(i) = \begin{cases} B & if\ i \leq l_r - l_0 \\ 0 & otherwise \end{cases}$
  - $a_{\diamond *}(i) = 0$
  - $a_{\diamond\dagger}(i) = 0$

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Examples of the policies – disability and age pension

- Disability and age pension pays a recurring benefit B_d from a period when an insured, current age l_0, becomes disabled and until a pre-agreed age l_r and thereafter a recurring benefit B_s (regardless if an insured was disabled or not) Policy functions have a form:
  - $a_*(i) = \begin{cases} -\pi & if\ i \leq l_r - l_0 - 1 \\ B_s & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_\diamond(i) = \begin{cases} B_d & if\ i \leq l_r - l_0 \\ B_s & otherwise \end{cases}$
  - $a_{*\dagger}(i) = 0$
  - $a_{*\diamond}(i) = 0$
  - $a_{\diamond *}(i) = 0$
  - $a_{\diamond\dagger}(i) = 0$

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Examples of the policies – disability and age pure endowment

- Disability and age pure endowment pays a one time benefit B_d when an insured, current age l_0, gets disabled before a preagreed age l_r, and a one time benefit B_s if an insured reaches age l_r(regardless if insured is disabled or not). Policy functions have a form:
  - $a_*(i) = \begin{cases} -\pi & if\ i \leq l_r - l_0 - 1 \\ B_s & if\ i = l_r - l_0 \\ 0 & otherwise \end{cases}$
  - $a_\dagger(i) = 0$
  - $a_\diamond(i) = 0$
  - $a_{*\dagger}(i) = 0$
  - $a_{*\diamond}(i) = \begin{cases} B_d & if\ i \leq l_r - l_0 \\ 0 & otherwise \end{cases}$
  - $a_{\diamond *}(i) = 0$
  - $a_{\diamond\dagger}(i) = 0$

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Expected present value revisited

- Even though we used more complicated model, we still use expected present value of the cashflow to model the policy.
- $V_0 = \sum_{i=0}^{\infty} d^i \left(\sum_{s \in S} (a_s(i) P(X_i = s) + \sum_{\substack{r \in S \\ r \neq s}} a_{r,s}(i) P(X_{i-1} = r, X_i = s)) \right)$
- Note however that it is more difficult to calculate the expected present value explicitly as we need to know probability of being in each state – more difficult for a general model
- In Life Insurance course you will learn about Thiele-equation which allows for calculating it recursively
- In this course, we will be just thinking about simulating Markov chain and computing MC-expectation

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Pricing and reserving of the general contracts

- Pricing – equivalence principle still applies, i.e. the fair premium gives an expected present value equal to 0
- Reserving at time 0 – we use the one-time premium and equivalence principle to calculate the mathematical reserve at time 0
- Note that the reserving at time n>0 is more complicated for the general model as we need to compute one-time premium paid at n such that the expected present value **conditioned on the state s,** in which the insured is at time n, is equal to 0. The reserve is therefore state-dependent!!!

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Example of reserving – disability and age pension

- $a_*(i) = \begin{cases} 0 & if\ i \le l_r - l_0 - 1 \\ B_s & otherwise \end{cases}$
- $a_\dagger(i) = 0$
- $a_\diamond(i) = \begin{cases} B_d & if\ i \le l_r - l_0 \\ B_s & otherwise \end{cases}$
- $a_{*\dagger}(i) = 0$
- $a_{*\diamond}(i) = 0$
- $a_{\diamond *}(i) = 0$
- $a_{\diamond\dagger}(i) = 0$

$$V_k\Big|_{X_k = s_k} = \sum_{i=0}^{\infty} d^i \left(\sum_{s \in S} (a_s(i) P(X_{k+i} = s \mid X_k = s_k) + \sum_{\substack{r \in S \\ r \neq s}} a_{r,s}(i) P(X_{k+i-1} = r, X_{k+i} = s \mid X_k = s_k)) \right)$$

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Life insurance

Unit-linked products

Classified: General Business

Traditional vs. unit-linked pension

- In traditional pension, the insured pays premiums as long as they are active, they know what kind of benefit they can expect after retiring, but do not know how long they are going to live. They are moreover a subject of risk sharing, i.e. they can get less than they paid in and the difference will cover the benefits of those living longer
- Those products are very risky for the insurance companies as:
  - Interest rate varies
  - People live longer and the real life expectancy can be longer than expected
- Therefore, more and more common to offer products resembling more the investment funds
  - The insured saves a premium in a chosen investment fund, which brings some profit over the years
  - No guarantee from the insurance company how high benefit one will get as a retired and the final pension depends only on the profit
  - The whole risk for low profit lies on the insured.
  - Often no risk sharing: unused funds go to the family, but if someone lives for “too long”, no pension when funds all used

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Pure unit-link product

- Premiums: periodical premiums, either fixed size or linked to the salary, paid until retirement age. But can also be just a one time premium
- Insurance company buys units of a chosen investment fund for the premium (minus some commission)
- If insured survives to the retirement, they get the money from premiums and return, either until death or until there is no more funds
- If the insured dies first, the rest of money paid out to the family

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Pricing and reserving

- The role of the insurance company in unit-linked products: a party who buys and holds investment funds in the name of the insured
- No real risk here, and therefore pricing is reduced to the calculation of commission which insurance company gets
- Reserving is also easy (at least for accounting): just put as a reserve the current value of the investment fund

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Are unit-linked product really offered?

- Yes:
  - As an element of public pension system ("innskuddspensjon") – then premiums linked to the salary and paid by the employer,
  - as a private saving plan (IPS = individuell pensjonsparing, or other saving plan) – then no fixed premium and one can pay as much as one wants
- Innskuddspensjon also combined with disability insurance

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Options and derivatives

"an insurance for financial markets"

Classified: General Business

Book, chapter 3, section 3.5, 3.7.1

Derivatives – what it is?

- Assume that we have an asset S, with initial value S_0, which can give an unknown return R over a period of time T (i.e. value at T is $S_T = S_0 * (1 + R)$).
- A (European) derivative is a contract which can give you a payout depending on the value of the return R. In mathematical term: payoff at T is equal to be $g(S_T)$, where g is a function.

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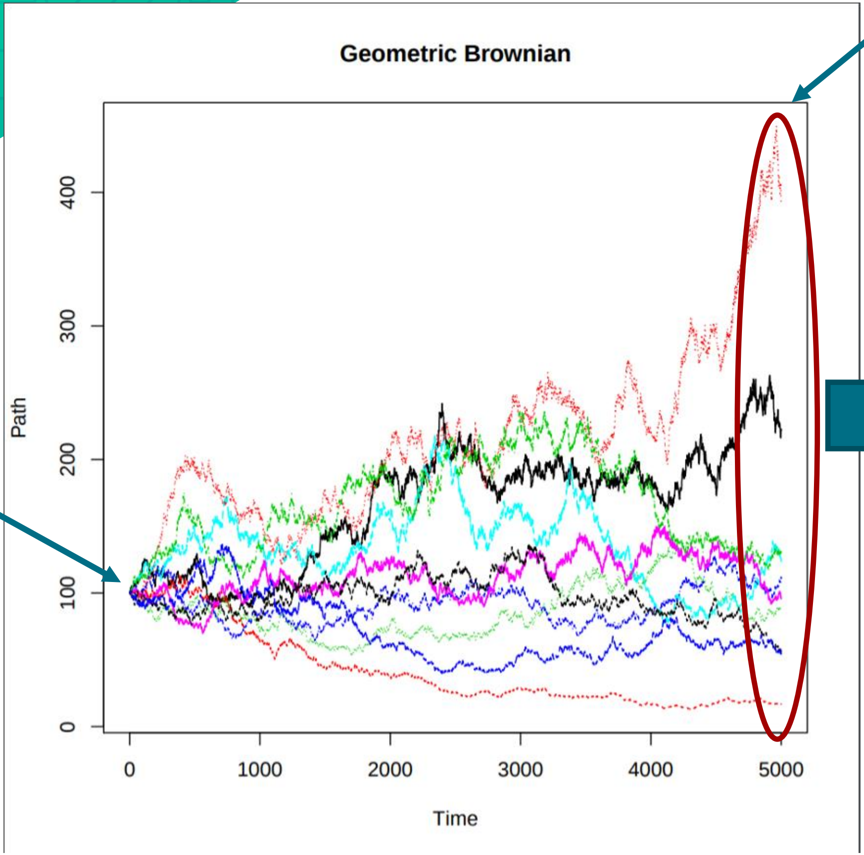
Derivative

$S_T = S_0 * (1 + R)$

S_0



Our derivative is paying us $g(S_T)$

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An European call options

An **European call option** gives you a right (but not an obligation) to buy an asset (e.g. stock or an index/portfolio) at a predefined time T (called maturity) for a predefined price K (called a strike).

A rational investor is going to use that option only if strike K is lower than the asset price at time T (because if the strike is higher than the market price, they can buy it for the market price instead of using the option)

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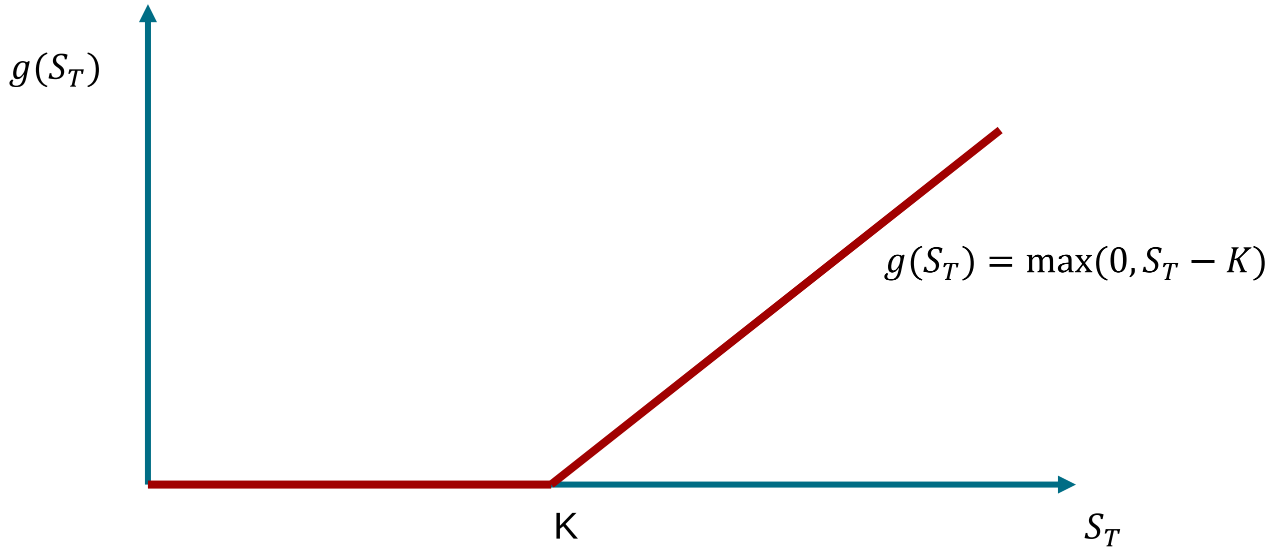
Function g for European call option



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An European put options

An **European put option** gives you a right (but not an obligation) to sell an asset (e.g. stock or an index/portfolio) at a predefined time T (called maturity) for a predefined price K (called a strike).

A rational investor is going to use that option only if strike K is higher than the asset price at time T (because if the strike is lower than the market price, they can sell it for the market price instead of using the option)

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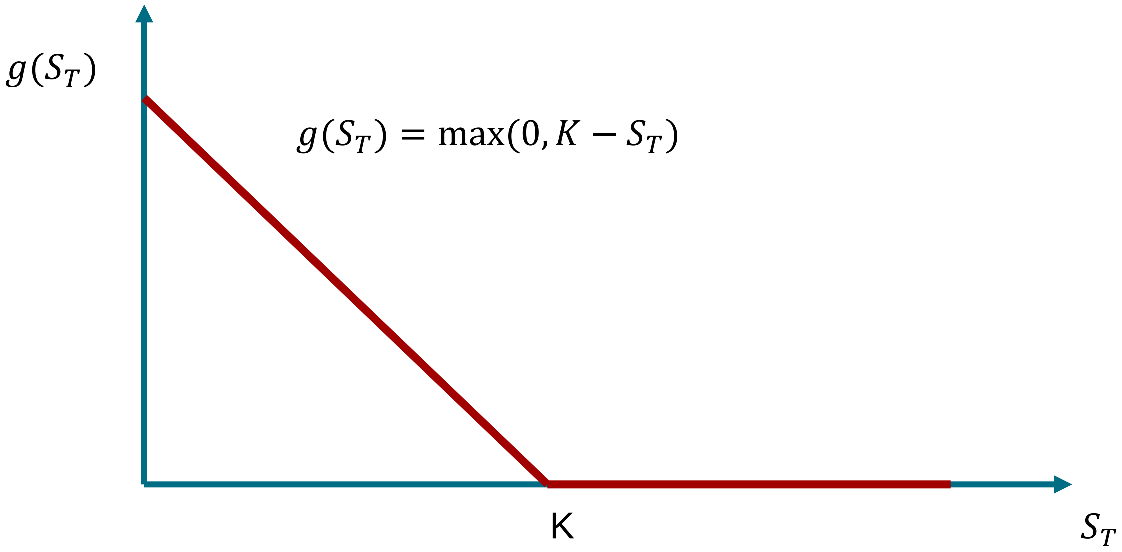
Function g for European put option



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Options vs. guaranteed return

- Let $r_g = \frac{K}{S_0} - 1$ and $R = \frac{S_T}{S_0} - 1$
- Note that for call option:

$$g(S_T) = \max(0, S_T - K) = \max\left(0, \frac{S_T}{S_0} - 1 - \left(\frac{K}{S_0} - 1\right)\right) S_0 = \max(0, R - r_g)\, S_0$$

- Note that for put option:

$$g(S_T) = \max(0, K - S_T) = \max\left(0, \frac{K}{S_0} - 1 - \left(\frac{S_T}{S_0} - 1\right)\right) S_0 = \max(0, r_g - R)\, S_0$$

- Interpretation: option can be seen as a guarantee of a return r_g

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How to use option for hedging?

- Hedging is an activity of reducing your financial risk by buying additional instruments, e.g. derivatives.
- Assume you own a unit of stock S, which is worth S_0 at time 0
- You additionally have a put option with strike K, which gives $r_g = \frac{K}{S_0} - 1$
- You know that S, including the option, will give you at least the return of r_g
- A put option is therefore an insurance against falling prices
- Similarly, a call option is an insurance against rising prices
- However, no risk sharing!!!!

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How to calculate value of the option?

- Lets assume that we know the distribution of the asset price at maturity S_T. Assume risk-free interest rate on bank count to be continuously compounded and equal to r.
- One could guess that the market price would be discounted expected cashflow, i.e. $E(g(S_T))e^{-rT}$. Note however, that such a price can introduce **arbitrage** (profit without risk)
- How? Assume g(x)=x. If $E\left(g(S_T)\right)e^{-rT} = E(S_T)e^{-rT}$ is different than S_0, then have a paradox: we have a market price S_0 and we claim that the market price should be $E(S_T)e^{-rT}$.
- This shows us that in order to avoid paradoxes and arbitrage, our model must have the property $E(S_T)e^{-rT} = S_0$.

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Risk neutral modelling of S

- Let $S_T = S_0 \exp\left(\mu T + \sigma\sqrt{T}\epsilon\right), where\ \epsilon \sim N(0,1)$.
- S_T/S_0 is log-normally distributed and therefore $ES_T = S_0 e^{\mu T + \frac{\sigma^2}{2}T}$
- In general, $ES_T e^{-rT} = S_0 e^{\mu T + \frac{\sigma^2}{2}T - rT} \neq S_0$
- Introduce however a new "**risk neutral model**", under risk neutral probability Q, such that

  $S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\epsilon\right), where\ \epsilon \sim N(0,1)$ "under Q".
- Then the risk neutral expected value (i.e. expected value under probability Q) is equal to:
- $E^Q(S_T e^{-rT}) = S_0 e^{(r-\frac{\sigma^2}{2})T + \frac{\sigma^2}{2}T - rT} = S_0$.
- We see that such a model does not introduce arbitrage or paradoxes and is market consistent (i.e. gives a price which is consistent with the price from the market)

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Why "risk-neutral"?

- A risk neutral investor does not distinguish between a risky and non-risky investment as long as they have exactly the same expected values.
- Easy to check that under Q-probability, expected value of the investment in S and in the bank account till be the same $S_0 e^{rT}$, therefore a risk-neutral investor won't distinguish between them.
- One can prove a general result that a risk-neutral valuation (under some conditions) leads to arbitrage free prices.

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Going back to the option pricing

- Now as we have a good model to price S, we use it to price our options
- Let V_0 be a price of a derivative with payoff function g, maturity T and under continuously compounded interest rate r. It can be proven that

$$V_0 = E^Q(g(S_T)e^{-rT}), where\ Q\ is\ a\ risk\ neutral\ probability$$

- How to deal with this guy?
  - Simulate S_T under risk neutral probability and compute the MC-average to approximate V_0
  - Or: Find an explicit formula for V_0
- How to prove it? Not trivial, requires good knowledge of martingale theory and stochastic processes/stochastic integration.

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Black-Scholes formula for put option

- Let $S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\epsilon\right), where\ \epsilon \sim N(0,1)$ "under risk neutral probability Q", where r is continuously compounded interest rate. Then a price of a put option with strike K and maturity T can be expressed as
- $V_0 = Ke^{-rT}\Phi(a) - S_0\Phi(a - \sigma\sqrt{T}),$
  $where\ \Phi\ is\ CDF\ for\ standard\ normal\ distribution\ and$

$$a = \frac{\ln\frac{K}{S_0} - rT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

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Black-Scholes formula for put option - proof

$$V_0 = E^Q(g(S_T)e^{-rT}) = E^Q(\max(0, K - S_T)\, e^{-rT})$$

$$= E^Q\left(\max\left(0, K - S_0\exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\epsilon\right)\right)e^{-rT}\right)$$

$$= \#\#\# \quad \phi - standard\ normal\ density\ \#\#\# =$$

$$= e^{-rT}\int_{-\infty}^{+\infty}\max\left(0, K - S_0\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}s\right)\right)\phi(s)ds =$$

$$= \#\#\# \quad definition\ of\ a\ and\ where\ maximim\ is\ non\ 0\ \#\#\# =$$

$$= e^{-rT}\int_{-\infty}^{a}\left(K - S_0\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}s\right)\right)\phi(s)ds =$$

$$= e^{-rT}K\int_{-\infty}^{a}\phi(s)ds - e^{-rT}\int_{-\infty}^{a}\left(S_0\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}s\right)\right)\phi(s)ds \ =$$

$$=: e^{-rT}K\Phi(a) - I$$

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Black-Scholes formula for put option – proof, cont.

$$I = e^{-rT}\int_{-\infty}^{a}\left(S_0 \exp\left(\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}s\right)\right)\phi(s)ds \;=$$

$$= e^{-rT}S_0\exp(\left(r-\frac{\sigma^2}{2}\right)T)\int_{-\infty}^{a}\exp(\sigma\sqrt{T}s)\,\phi(s)ds$$

$$= S_0\exp(\left(-\frac{\sigma^2}{2}\right)T)\int_{-\infty}^{a}\exp(\sigma\sqrt{T}s)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}s^2}ds$$

$$= S_0\exp(\left(-\frac{\sigma^2}{2}\right)T)\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(s-\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2 T}ds$$

$$= S_0\exp\left(\left(-\frac{\sigma^2}{2}\right)T\right)\exp(0.5*\sigma^2T)\int_{-\infty}^{a}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(s-\sigma\sqrt{T})^2}ds$$

$$= S_0\exp\left(\left(-\frac{\sigma^2}{2}\right)T\right)\exp(0.5*\sigma^2T)\int_{-\infty}^{a-\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(u)^2}du \;= S_0\Phi(a-\sigma\sqrt{T}),$$

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Black Scholes for guaranteed return

- $r_g = \frac{K}{S_0} - 1$ *therefore:*
- $V_0 = \left(\left(1 + r_g\right)e^{-rT}\Phi(a) - \Phi\left(a - \sigma\sqrt{T}\right)\right) S_0$
- $a = \frac{\ln\left(1+r_g\right) - rT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$

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Call option pricing

- Easily one can redo the proof for call option
- Or one can use **call-put option parity** for call and put options with the same strike and maturity
- Note that $\max(0, S_T - K) - \max(0, K - S_T) = S_T - K$
- Therefore $\mathrm{E}^{\mathrm{Q}}(\max(0, S_T - K) - \max(0, K - S_T))e^{-rT} = \mathrm{E}^{\mathrm{Q}}(S_T - K)e^{-rT}$
- Knowing price V_0^P for put and the fact that Q is a risk neutral measure one gets that the price V_0^C for call is equal
- $V_0^C = \mathrm{V}_0^{\mathrm{P}} + \mathrm{S}_0 - \mathrm{K}e^{-\mathrm{rT}}$

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More on Black-Scholes model

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“Book, chapter 3, section 3.7.2

How option valuation really looks like

- ”vanilla” options (i.e. plain options like call and put options) are traded on the market with a proper market price
  - no great need for a model as you can just check market prices
  - Model is only necessary to model the future development of market price of an option, given the changes in the underlying instrument S
- “exotic” options (often complicated custom-made options) are not widely traded on the market and one needs a model to come up with a price
  - E.g. Asian option which depend on the average price of the underlying instrument S, instead of price at time maturity
  - Model is necessary both for time t=0 and for projections of future price changes
- How one does the valuation of an exotic option?
  - One chooses a model (e.g. B-S model), **calibrates** it so that the model reproduces the market prices for the vanilla instruments (e.g. call and put options) and then uses $E^Q(g(S_T)e^{-rT})$ to valuate the exotic option

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Black-Scholes (B-S) formula for put option, again

- Let $S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}\epsilon\right), where\ \epsilon \sim N(0,1)$ "under risk neutral probability Q", where r is continuously compounded interest rate. Then a price of a put option with strike K and maturity T can be expressed as
- $V_0 = Ke^{-rT}\Phi(a) - S_0\Phi(a - \sigma\sqrt{T}),$
  $where\ \Phi\ is\ CDF\ for\ standard\ normal\ distribution\ and$

$$a = \frac{\ln\frac{K}{S_0} - rT + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$$

- We will first investigate the assumption of the B-S model and then talk more about the options

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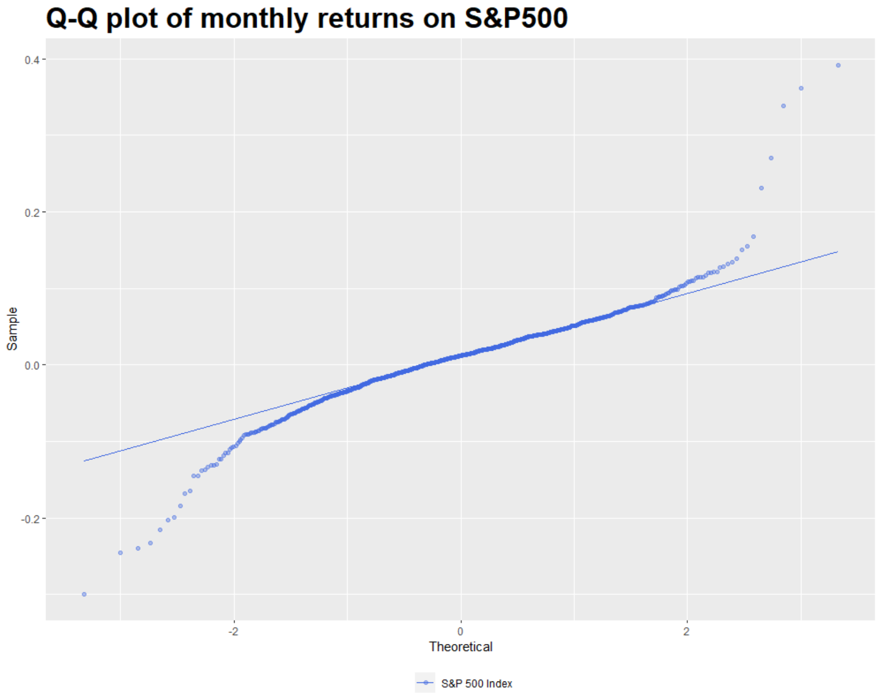
Monthly logreturns on S&P500 Index



0.05-quantile	0.25-quantile	median	mean	0.75-quantile	0.95-quantile	St.dev
-7.6%	-1,6%	1,2%	0,9%	3,8%	8.0%	5,4%

- Clear sign of non-normality
- Negative skew (i.e. distribution is assymmetric and left tail is thicker and median is higher than mean)
- Fat tails

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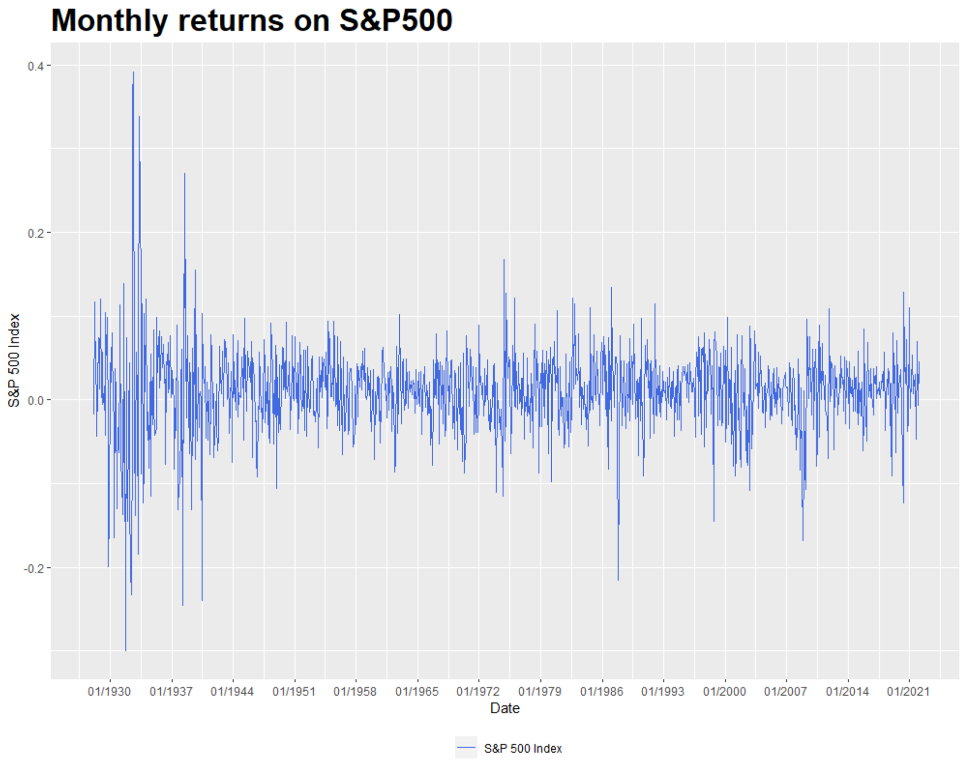
Monthly logreturns on S&P500 Index



Characteristics:

- Non-constant and stochastic volatility
- Volatility clustering, i.e. one observes periods of both low and high volatility
- Leverage effect, i.e. high volatility coincides with negative returns

Class…

Problems with Black-Scholes model

- Assumes normal logreturns – we have seen that this is not the case – no fat tails, no skewness, constant volatility
- Call and put options are commonly traded – there is a good market price on many stocks. B-S fails to predict those market prices of call and puts.
- Do people believe in B-S? **No!!!** Other much better models are used to price the option.
- But is B-S used? **Yes!** But not as a model to explain/predict market prices but to “quote” market prices (i.e. when a derivative broker gives a price of the option, they use B-S parameters)

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What the call/put option really is?

- Note that B-S formula has four parameters: S_0, σ, r, K. Three parameters are known very well S_0, r, K. So, if we believe in the B-S formula, the option really depends only on volatility σ.
- This is why it can be thought that the call/put option is mostly a bet on volatility – if volatility will increase in the future, the price of call/put option will go up.
- This reflected in how the options are traded:
  - If you a trader to buy an option, they will give you a price (*"give you a quote"*) in terms of volatility σ, not a price V_0.
  - In other words: You will get volatility σ and in order to calculate price in NOK, you use B-S formula. That is why it is said that B-S formula is used for quoting the price
- The volatility σ_{imp} which gives exactly the same price for an option as price observed at a market is called **implied volatility**

Volatility smile/skew

- So far we talked about one option for an underlying instrument. However, in practice, we have many options for different strikes and maturities for the same underlying.
- We have therefore different market prices for those options, and we can use the B-S formula to find implied volatilities.
- We will therefore have a curve: $K \mapsto \sigma_{imp,K}$ and $T \mapsto \sigma_{imp,T}$. If B-S model really holds, those curves should be flat.
- However, they aren't. We have so-called volatility skew and/or smile

- This is also a reason why no one really uses B-S for pricing of the exotic options: the model is not able to reproduce market prices for vanilla options and there is therefore no reason why they would do better with exotic options

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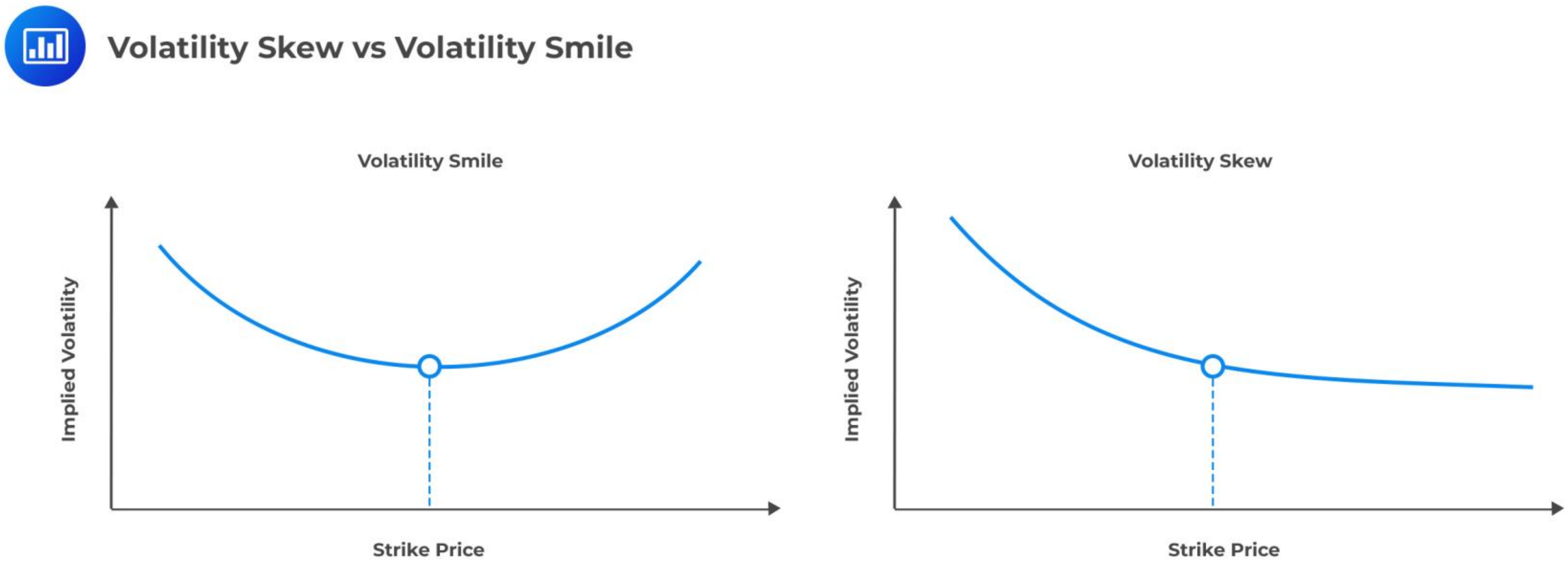
Volatility smile/skew



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More on derivatives

- Greeks: the sensitivity of the derivative price on the parameters of the B-S formula
  - Delta: derivative of V_0 with respect to S_0
  - Theta: derivative of V_0 with respect to time to maturity T
  - Vega: derivative of V_0 with respect to time to maturity σ
  - Rho: derivative of V_0 with respect to time to maturity r
- Greeks are super important and used. E.g. a trader who buys and sells option will try to have delta of the whole portfolio to be equal 0, and then the value of portfolio is not affected by the small changes of the underlying stock price

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Calculation of Greeks

Lemma:

$$Ke^{-rT}\phi(a) = S_0\phi(a - \sigma\sqrt{T}), \qquad where\ \phi - density\ of\ N(0,1)$$

Proof:

$$\ln\left(\frac{\phi(a-\sigma\sqrt{T})}{\phi(a)}\right) = \ln\left(\frac{\exp(-0.5(a-\sigma\sqrt{T})^2)}{\exp(-0.5a^2)}\right) = -0.5 * \left((a-\sigma\sqrt{T})^2 - a^2\right) =$$

$$= -0.5(a - \sigma\sqrt{T} - a)(a - \sigma\sqrt{T} + a) = 0.5\sigma\sqrt{T}(2a - \sigma\sqrt{T}) = \sigma\sqrt{T}a - 0.5\sigma^2 T =$$

$$= \sigma\sqrt{T}\frac{\ln\left(\frac{K}{S_0}\right) - rT + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} - 0.5\sigma^2 T = \ln\left(\frac{K}{S_0}\right) - rT$$

We take exponent to see that

$$\frac{\phi(a-\sigma\sqrt{T})}{\phi(a)} = \frac{K}{S_0}e^{-rT}$$

We get the lemma by rearranging terms.

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Calculation of Greeks

Delta for a put option

$$\Delta = \frac{\partial V_0^P}{\partial S_0} = -\Phi\left(a - \sigma\sqrt{T}\right)$$

Proof:

$$\frac{\partial V_0^P}{\partial S_0} = \frac{\partial}{\partial S_0}\left(Ke^{-rT}\Phi(a) - S_0\Phi(a - \sigma\sqrt{T})\right) = Ke^{-rT}\phi(a)\frac{\partial}{\partial S_0}a - \Phi\left(a - \sigma\sqrt{T}\right) - S_0\Phi\left(a - \sigma\sqrt{T}\right)\frac{\partial}{\partial S_0}a =$$

$$= \frac{\partial}{\partial S_0}a * \left(Ke^{-rT}\phi(a) - S_0\Phi\left(a - \sigma\sqrt{T}\right)\right) - \Phi\left(a - \sigma\sqrt{T}\right) = 0 - \Phi\left(a - \sigma\sqrt{T}\right) = -\Phi\left(a - \sigma\sqrt{T}\right).$$

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Options in life-insurance

I.e. why we really had a previous section of the course

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Pension with profit participation and unit-linked with guarantees

- We already learned about many pension products: defined benefit pension, unit linked products.
- There is however two more products, both including some kind of option/guarantee element: pensions with profit participation and unit-linked products with guarantees
- Remember about technical interest rate vs. fair valuation of life-insurance reserves? For products with option-elements, the fair valuation (i.e. market consistent valuations), those options needs to be separated and valued using an option valuation model, e.g. B-S. This is why we are talking about options!!!

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Unit linked with guarantees

- In a typical unit linked product with guarantees, an insurance company guarantees that at the retirement a retired will get at least K-amount of money, even if the value of the investment fund falls under K.
- In other words, the product includes a put option with strike K.
- The details of the option depends on the product and may include e.g. specification what happens when an insured dies before the retirement age.
- Product not really offered in Norway, as far as I know.

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Pension with profit participation

- Much more complicated product, used to be popular as a pension from employer (tjenestepensjon) in private sector.
- The insured has an "account", where the premiums are paid to by an employer. The funds are invested by the insurance company and the insurance company guarantees that the investment will give at least a guaranteed **yearly** return r_g each year.
- If the return in a particular year is higher than r_g, a part of the exceeding amount goes to the account, a part goes to a **buffer account**, and a part goes as a profit to the insurance company
- If the return is lower than r_g, then the difference is first covered from the buffer account, and if there is not sufficient funds there, the insurance company covers the loss from its own funds
- Here we have a sequence of yearly guarantees/options. Also need to be valued as an option

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