Solutions to exercises - Week 35

Transformations of univariate random variables:

- Exercises 2.1 and 2.6.a

Moment generating functions:

- Exercises 2.30 and 2.33

Miscellaneous:

- Exercises 3.13 and 3.17

Location and scale families:

- Exercises 3.38 and 3.39

Exercise 2.1.a (a direct argument)

$$f_X(x) = 42x^5(1-x) \quad \text{for } 0 < x < 1$$

$$Y = g(X) = X^3$$

For $y \in \mathcal{Y} = (0,1)$ the cdf of Y is obtained by

$$F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3}) = F_X(y^{1/3})$$

Then for $y \in \mathcal{Y} = (0,1)$ the pdf is given by

$$f_Y(y) = F_Y'(y) = F_X'(y^{1/3})\frac{1}{3}y^{-2/3} = f_X(y^{1/3})\frac{1}{3}y^{-2/3}$$

$$= 42y^{5/3}(1-y^{1/3})\frac{1}{3}y^{-2/3} = 14y(1-y^{1/3})$$

Exercise 2.1.a (using Theorem 2.1.5)

$$f_X(x) = 42x^5(1-x) \quad \text{for } 0 < x < 1$$

$$Y = g(X) = X^3$$

$$g^{-1}(y) = y^{1/3} \qquad \frac{d}{dy}g^{-1}(y) = \frac{1}{3}y^{-2/3}$$

For $y \in \mathcal{Y} = (0,1)$ the pdf of Y is given by

$$f_Y(y) = f_X(y^{1/3})\frac{1}{3}y^{-2/3}$$

$$= 42y^{5/3}(1-y^{1/3})\frac{1}{3}y^{-2/3}$$

$$= 14y(1-y^{1/3})$$

Exercise 2.1.b

$$f_X(x) = 7e^{-7x} \quad \text{for } x > 0$$

$$Y = g(X) = 4X + 3$$

$$g^{-1}(y) = \frac{1}{4}(y-3) \qquad \frac{d}{dy}g^{-1}(y) = \frac{1}{4}$$

For $y \in \mathcal{Y} = (3,\infty)$ the pdf of Y is given by

$$f_Y(y) = f_X\left(\frac{y-3}{4}\right)\frac{1}{4}$$

$$= 7e^{-7(x-3)/4}\frac{1}{4}$$

$$= \frac{7}{4}e^{-(7/4)(x-3)}$$

Exercise 2.1.c

$$f_X(x) = 30x^2(1-x)^2 \quad \text{for } 0 < x < 1$$

$$Y = g(X) = X^2$$

$$g^{-1}(y) = y^{1/2} \qquad \frac{d}{dy} g^{-1}(y) = \frac{1}{2} y^{-1/2}$$

For $y \in \mathcal{Y} = (0,1)$ the pdf of Y is given by

$$f_Y(y) = f_X(y^{1/2}) \frac{1}{2} y^{-1/2}$$

$$= 30 y^{2/2}(1-y^{1/2})^2 \frac{1}{2} y^{-1/2}$$

$$= 15 y^{1/2}(1-y^{1/2})^2$$

Exercise 2.6.a (a direct argument)

$$f_X(x) = \frac{1}{2} e^{-|x|} \quad \text{for} \quad -\infty < x < \infty$$

$$Y = g(X) = |X|^3$$

For $y \in \mathcal{Y} = (0,\infty)$ the cdf of Y is obtained by

$$F_Y(y) = P(Y \le y) = P(|X|^3 \le y)$$

$$= P(|X| \le y^{1/3})$$

$$= P(-y^{1/3} \le X \le y^{1/3})$$

$$= F_X(y^{1/3}) - F_X(-y^{1/3})$$

Then for $y \in \mathcal{Y} = (0,\infty)$ the pdf is given by

$$f_Y(y) = F_Y'(y)$$

$$= F_X'(y^{1/3}) \frac{1}{3} y^{-2/3} - F_X'(-y^{1/3}) \frac{-1}{3} y^{-2/3}$$

$$= f_X(y^{1/3}) \frac{1}{3} y^{-2/3} + f_X(-y^{1/3}) \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{2} e^{-|y^{1/3}|} \frac{1}{3} y^{-2/3} + \frac{1}{2} e^{-|-y^{1/3}|} \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{3} y^{-2/3} e^{-y^{1/3}}$$

Exercise 2.30.a

$$f_X(x) = \frac{1}{c} \quad \text{for} \quad 0 < x < c$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_0^c e^{tx} \frac{1}{c}\,dx$$

$$= \frac{1}{c}\left[\frac{1}{t} e^{tx}\right]_0^c$$

$$= \frac{e^{tc} - 1}{ct}$$

Exercise 2.30.b

$$f_X(x) = \frac{2x}{c^2} \quad \text{for} \quad 0 < x < c$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_0^c e^{tx} \frac{2x}{c^2}\,dx$$

$$= \left[\frac{1}{t} e^{tx} \frac{2x}{c^2}\right]_0^c - \int_0^c \frac{1}{t} e^{tx} \frac{2}{c^2}\,dx \qquad \text{(integration-by-parts)}$$

$$= \frac{2}{ct} e^{tc} - \left[\frac{2}{c^2 t^2} e^{tx}\right]_0^c$$

$$= \frac{2}{c^2 t^2}\left(cte^{tc} - e^{tc} + 1\right)$$

Exercise 2.30.c

$$f_X(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta} \quad \text{for} \quad -\infty < x < \infty$$

For $-1/\beta < t < 1/\beta$ we have that

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\beta} e^{-|x-\alpha|/\beta}\,dx$$

$$= \int_{-\infty}^{\alpha} e^{tx} \frac{1}{2\beta} e^{(x-\alpha)/\beta}\,dx + \int_{\alpha}^{\infty} e^{tx} \frac{1}{2\beta} e^{-(x-\alpha)/\beta}\,dx$$

$$= \frac{1}{2\beta} e^{-\alpha/\beta} \int_{-\infty}^{\alpha} e^{(t+1/\beta)x}\,dx + \frac{1}{2\beta} e^{\alpha/\beta} \int_{\alpha}^{\infty} e^{(t-1/\beta)x}\,dx$$

$$= \frac{1}{2\beta} e^{-\alpha/\beta} \left[\frac{1}{t+1/\beta} e^{(t+1/\beta)x}\right]_{-\infty}^{\alpha} + \frac{1}{2\beta} e^{\alpha/\beta} \left[\frac{1}{t-1/\beta} e^{(t-1/\beta)x}\right]_{\alpha}^{\infty}$$

$$= \frac{1}{2\beta} e^{-\alpha/\beta} \frac{1}{t+1/\beta} e^{(t+1/\beta)\alpha} - \frac{1}{2\beta} e^{\alpha/\beta} \frac{1}{t-1/\beta} e^{(t-1/\beta)\alpha}$$

$$= \frac{e^{\alpha t}(t-1/\beta) - e^{\alpha t}(t+1/\beta)}{2\beta(t+1/\beta)(t-1/\beta)} = \frac{-2e^{\alpha t}/\beta}{2\beta\left(t^2 - 1/\beta^2\right)}$$

$$= \frac{e^{\alpha t}}{1-\beta^2 t^2}$$

Exercise 2.30.d

$$f_X(x) = P(X = x) = \binom{r+x-1}{x} p^r (1-p)^x \qquad \text{for} \quad x = 0, 1, \ldots$$

Note first that

$$\sum_{x=0}^{\infty} \binom{r+x-1}{x} \left(1-(1-p)e^t\right)^r \left((1-p)e^t\right)^x = 1 \quad \text{if} \quad (1-p)e^t < 1$$

Therefore (when $t < -\log(1-p)$)

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} f_X(x) = \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x$$

$$= p^r \sum_{x=0}^{\infty} \binom{r+x-1}{x} \left[(1-p)e^t\right]^x$$

$$= \left(\frac{p}{1-(1-p)e^t}\right)^r$$

Exercise 2.33.a

$$P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda} \quad \text{for} \quad x=0,1,\dots$$

$$M_X(t)=\sum_{x=0}^{\infty} e^{tx} f_X(x) = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x}{x!}e^{-\lambda}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

$$R_X(t)=\log M_X(t)=\lambda(e^t-1)$$

$$R_X'(t)=\lambda e^t \qquad R_X''(t)=\lambda e^t$$

$$\mathrm{E}X = R_X'(0)=\lambda$$

$$\mathrm{Var}\, X = R_X''(0)=\lambda$$

Exercise 2.33.b

$$P(X=x)=p(1-p)^x \quad \text{for} \quad x=0,1,\dots$$

$$M_X(t)=\sum_{x=0}^{\infty} e^{tx} f_X(x) = \sum_{x=0}^{\infty} e^{tx} p(1-p)^x$$

$$= p\sum_{x=0}^{\infty}\left((1-p)e^t\right)^x = \frac{p}{1-(1-p)e^t} \quad \text{when } (1-p)e^t<1$$

$$R_X(t)=\log M_X(t)=\log p-\log\{1-(1-p)e^t\}$$

$$R_X'(t)=\frac{(1-p)e^t}{1-(1-p)e^t} \qquad R_X''(t)=\frac{(1-p)e^t\left(1-(1-p)e^t\right)+\left((1-p)e^t\right)^2}{\left(1-(1-p)e^t\right)^2}$$

$$\mathrm{E}X = R_X'(0)=\frac{1-p}{p}$$

$$\mathrm{Var}\, X = R_X''(0)=\frac{(1-p)p+(1-p)^2}{p^2}=\frac{1-p}{p^2}$$

Exercise 2.33.c

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)} \quad \text{for} \quad -\infty<x<\infty$$

$$M_X(t)=\int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/(2\sigma^2)}\,dx$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-[x^2-2x(\mu+\sigma^2 t)+\mu^2]/(2\sigma^2)}\,dx$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-[\{x-(\mu+\sigma^2 t)\}^2-(\mu+\sigma^2 t)^2+\mu^2]/(2\sigma^2)}\,dx$$

$$=e^{-[(\mu+\sigma^2 t)^2-\mu^2]/(2\sigma^2)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\{x-(\mu+\sigma^2 t)\}^2/(2\sigma^2)}\,dx$$

$$=e^{-[(\mu+\sigma^2 t)^2-\mu^2]/(2\sigma^2)} = e^{\mu t+\sigma^2 t^2/2}$$

$$R_X(t)=\log M_X(t)=\mu t+\sigma^2 t^2/2$$

$$R_X'(t)=\mu+\sigma^2 t$$

$$R_X''(t)=\sigma^2$$

$$\mathrm{E}X = R_X'(0)=\mu$$

$$\mathrm{Var}\, X = R_X''(0)=\sigma^2$$

Exercise 3.13

Discrete random variable X with range $\mathcal{X} = \{0,1,2,\ldots.\}$

The 0-truncated random variable X_T has pmf

$$P(X_T = x) = P(X = x | X > 0)$$

$$= \frac{P\big((X = x) \cap (X > 0)\big)}{P(X > 0)}$$

$$= \frac{P(X = x)}{P(X > 0)}$$

for $x = 1, 2, \ldots.$

We have ($n \geq 1$)

$$\mathrm{E}X_T^n = \sum_{x=1}^{\infty} x^n P(X_T = x) = \sum_{x=1}^{\infty} x^n \frac{P(X = x)}{P(X > 0)}$$

$$= \frac{1}{P(X > 0)} \sum_{x=0}^{\infty} x^n P(X = x) = \frac{\mathrm{E}X^n}{P(X > 0)}$$

Hence

$$\mathrm{E}X_T = \frac{\mathrm{E}X}{P(X > 0)}$$

$$\mathrm{Var}\, X_T = \mathrm{E}X_T^2 - (\mathrm{E}X_T)^2 = \frac{\mathrm{E}X^2}{P(X > 0)} - \left(\frac{\mathrm{E}X}{P(X > 0)}\right)^2$$

$$= \frac{\mathrm{Var}\, X + (\mathrm{E}X)^2}{P(X > 0)} - \left(\frac{\mathrm{E}X}{P(X > 0)}\right)^2$$

a) $X \sim \mathrm{Poisson}(\lambda) \qquad \mathrm{E}X = \mathrm{Var}\, X = \lambda$

$$P(X > 0) = 1 - P(X = 0) = 1 - e^{-\lambda}$$

$$\mathrm{E}X_T = \frac{\lambda}{1 - e^{-\lambda}} \qquad \mathrm{Var}\, X_T = \frac{\lambda + \lambda^2}{1 - e^{-\lambda}} - \left(\frac{\lambda}{1 - e^{-\lambda}}\right)^2$$

b) $X \sim \text{negative binomial}(r, p)$

$$\mathrm{E}X = \frac{r(1-p)}{p} \qquad \mathrm{Var}\, X = \frac{r(1-p)}{p^2}$$

$$P(X > 0) = 1 - P(X = 0) = 1 - p^r$$

$$\mathrm{E}X_T = \frac{r(1-p)}{p(1-p^r)}$$

$$\mathrm{Var}\, X_T = \frac{r(1-p) + r^2(1-p)^2}{p^2(1-p^r)} - \left(\frac{r(1-p)}{p(1-p^r)}\right)^2$$

Exercise 3.17

$X \sim gamma(\alpha, \beta)$

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad \text{for } x > 0$$

$$\mathrm{E}X^\nu = \int_{-\infty}^{\infty} x^\nu f_X(x)\, dx = \int_0^\infty x^\nu \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\nu+\alpha-1} e^{-x/\beta}\, dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \beta^{\nu+\alpha} \Gamma(\nu + \alpha)$$

$$= \beta^\nu \frac{\Gamma(\nu + \alpha)}{\Gamma(\alpha)}$$

Exercise 3.38

$Z \sim f(z) \qquad \alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} f(z)\,dz$

X has pdf $\quad f_X(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$

Then

$$\alpha = P(X > x_\alpha) = \int_{x_\alpha}^{\infty} f_X(x)\,dx = \int_{x_\alpha}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx = \int_{\frac{x_\alpha - \mu}{\sigma}}^{\infty} f(z)\,dz$$

Therefore

$\frac{x_\alpha - \mu}{\sigma} = z_\alpha$ and $x_\alpha = \sigma z_\alpha + \mu$

Exercise 3.39

$$f_X(x) = \frac{1}{\sigma\pi} \frac{1}{1 + \left((x-\mu)/\sigma\right)^2} \quad \text{for} \quad -\infty < x < \infty$$

The pdf of $Z = (X-\mu)/\sigma$ is given by

$$f(z) = \frac{1}{\pi} \frac{1}{1+z^2} \quad \text{for} \quad -\infty < z < \infty$$

Recall that $\frac{d}{du} \arctan(u) = \frac{1}{1+u^2}$

The cdf of Z becomes

$$F(z) = \int_{-\infty}^{z} \frac{1}{\pi} \frac{1}{1+u^2}\,du = \frac{1}{\pi}\left[\arctan(u)\right]_{-\infty}^{z} = \frac{1}{\pi}\arctan(z) + \frac{1}{2}$$

We have

$$P(Z \le 0) = F(0) = \frac{1}{\pi}\arctan(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

Further we have

$$P(Z \le -1) = F(-1) = \frac{1}{\pi}\arctan(-1) + \frac{1}{2} = \frac{1}{\pi}\left(-\frac{\pi}{4}\right) + \frac{1}{2} = \frac{1}{4}$$

$$P(Z \ge 1) = 1 - F(1) = \frac{1}{2} - \frac{1}{\pi}\arctan(1) = \frac{1}{2} - \frac{1}{\pi}\frac{\pi}{4} = \frac{1}{4}$$

Finally for $X = \sigma Z + \mu$ we have

$$P(X \le x) = P(Z \le (x-\mu)/\sigma) = F((x-\mu)/\sigma)$$

$$P(X \le \mu) = F(0) = 1/2$$
$$P(X \le \mu - \sigma) = F(-1) = 1/4$$
$$P(X \ge \mu + \sigma) = 1 - P(X \le \mu + \sigma) = 1 - F(1) = 1/4$$