Solutions to exercises - Week 36

Beta function:

- Additional exercise

Exponential family of distributions:

- Exercises 3.28b-d, 3.30b and 3.33b-c

Bivariate distributions:

- Exercises 4.4a-b and 4.5

Additional exercise

We have the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

and the beta function

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx$$

We will prove the relation

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

We have to prove that

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta)$$

Now we may write

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

$$= \int_0^\infty \int_0^\infty x^{\alpha-1} e^{-x}\, y^{\beta-1} e^{-y} dydx$$

$$= \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} e^{-(x+y)}\, dydx$$

We then perform a change of variables:

$$u = x + y \quad \text{and} \quad v = \frac{x}{x+y}$$

Note that $u > 0$ and $0 < v < 1$

This gives $x = uv$ and $y = u(1-v)$

The Jacobian becomes

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -vu - (1-v)u = -u$$

We then obtain

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^1 (uv)^{\alpha-1}[u(1-v)]^{\beta-1} e^{-u}\, |-u|\, dvdu$$

$$= \int_0^\infty \int_0^1 u^{\alpha+\beta-1} e^{-u}\, v^{\alpha-1}(1-v)^{\beta-1}\, dvdu$$

$$= \int_0^\infty u^{\alpha+\beta-1} e^{-u} du \int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\, dv \;= \Gamma(\alpha+\beta)B(\alpha,\beta)$$

Exercise 3.28b

$$f(x|\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$$

For all x we may write:

$$f(x|\alpha,\beta)=\underbrace{I_{\{x>0\}}(x)}_{h(x)}\underbrace{\frac{1}{\beta^\alpha\Gamma(\alpha)}}_{c(\alpha,\beta)}\exp\left\{\underbrace{(\alpha-1)}_{w_1(\alpha,\beta)}\underbrace{\log x}_{t_1(x)}+\underbrace{\left(-\frac{1}{\beta}\right)}_{w_2(\alpha,\beta)}\underbrace{x}_{t_2(x)}\right\}$$

Exercise 3.28c

$$f(x|\alpha,\beta)=\begin{cases}\dfrac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0<x<1\\ 0 & \text{otherwise}\end{cases}$$

For all x we may write:

$$f(x|\alpha,\beta)=\underbrace{I_{(0,1)}(x)}_{h(x)}\underbrace{\frac{1}{B(\alpha,\beta)}}_{c(\alpha,\beta)}\exp\{\underbrace{(\alpha-1)}_{w_1(\alpha,\beta)}\underbrace{\log x}_{t_1(x)}+\underbrace{(\beta-1)}_{w_2(\alpha,\beta)}\underbrace{\log(1-x)}_{t_2(x)}\}$$

Exercise 3.28d

$$f(x|\lambda)=\begin{cases}\dfrac{\lambda^x}{x!}e^{-\lambda} & \text{for } x=0,1,2,....\\ 0 & \text{otherwise}\end{cases}$$

For all x we may write:

$$f(x|\lambda)=\underbrace{I_{\{0,1,2,....\}}(x)\frac{1}{x!}}_{h(x)}\underbrace{e^{-\lambda}}_{c(\lambda)}\exp\{\underbrace{\log(\lambda)}_{w_1(\lambda)}\underbrace{x}_{t_1(x)}\}$$

Exercise 3.30b

From exercise 3.28.d we have:

$$f(x|\lambda)=\underbrace{I_{\{0,1,2,....\}}(x)\frac{1}{x!}}_{h(x)}\underbrace{e^{-\lambda}}_{c(\lambda)}\exp\{\underbrace{\log(\lambda)}_{w_1(\lambda)}\underbrace{x}_{t_1(x)}\}$$

The relations in Theorem 3.4.2 become

$$\mathrm{E}\left(\frac{1}{\lambda}X\right)=1 \qquad \mathrm{Var}\left(\frac{1}{\lambda}X\right)=0-\mathrm{E}\left(-\frac{1}{\lambda^2}X\right)$$

From these relations we obtain

$$\mathrm{E}\,X=\lambda$$

$$\mathrm{Var}(X)=\lambda^2\left\{0+\frac{1}{\lambda^2}\lambda\right\}=\lambda$$

Exercise 3.33.b

Assume $X \sim n(\theta, a\theta^2)$

The pdf takes the form

$$f(x\,|\,\theta) = \frac{1}{\sqrt{2\pi}\sqrt{a\theta^2}} \exp\left(-\frac{(x-\theta)^2}{2a\theta^2}\right) \qquad a > 0 \ \text{ known}$$

This may be written

$$f(x\,|\,\theta) = \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{a\theta^2}} \exp\left(-\frac{1}{2a}\right)}_{c(\theta)} \exp\left(\underbrace{\frac{1}{2a\theta^2}}_{w_1(\theta)} \underbrace{\left(-x^2\right)}_{t_1(x)} + \underbrace{\frac{1}{a\theta}}_{w_2(\theta)} \underbrace{x}_{t_2(x)}\right)$$

For the normal distribution the full parameter space is

$$\left\{(\mu, \sigma^2) : -\infty < \mu < \infty,\ \sigma^2 > 0\right\}$$

Here we have a curved exponential distribution with parameter space

$$\left\{(\mu, \sigma^2) : \mu = \theta,\ \sigma^2 = a\theta^2,\ -\infty < \theta < \infty\right\}$$

Exercise 3.33.c

Assume $X \sim \text{gamma}(\alpha, 1/\alpha)$

The pdf takes the form

$$f(x\,|\,\alpha) = \begin{cases} \dfrac{\alpha^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

For all x we may write:

$$f(x\,|\,\alpha) = \underbrace{I_{\{x>0\}}(x)}_{h(x)} \underbrace{\frac{\alpha^\alpha}{\Gamma(\alpha)}}_{c(\alpha)} \exp\left\{\underbrace{(\alpha-1)}_{w_1(\alpha)} \underbrace{\log x}_{t_1(x)} + \underbrace{\alpha}_{w_2(\alpha)} \underbrace{(-x)}_{t_2(x)}\right\}$$

For the gamma distribution the full parameter space is

$$\left\{(\alpha, \beta) : \alpha > 0,\ \beta > 0\right\}$$

Here we have a curved exponential distribution with parameter space

$$\left\{(\alpha, \beta) : \alpha > 0,\ \beta = 1/\alpha\right\}$$

Exercise 4.4.a

$$f(x,y)=\begin{cases} C(x+2y) & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

We have:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^2 C(x+2y)\,dx\,dy = C\int_0^1\left[\frac{x^2}{2}+2yx\right]_0^2 dy$$

$$= C\int_0^1 (2+4y)\,dy = C\left[2y+2y^2\right]_0^1 = 4C$$

From this we obtain $C=\frac{1}{4}$

Exercise 4.4.b

$$f(x,y)=\begin{cases} \frac{1}{4}(x+2y) & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Marginal distribution of X (for $0<x<2$)

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{1}{4}(x+2y)\,dy$$

$$= \left[\frac{1}{4}(xy+y^2)\right]_0^1 = \frac{1}{4}(x+1)$$

Exercise 4.5.a

$$f(x,y)=\begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$P(X>\sqrt{Y}) = P(Y<X^2) = \int_0^1\int_0^{x^2}(x+y)\,dy\,dx$$

$$= \int_0^1\left[xy+\frac{y^2}{2}\right]_0^{x^2} dx = \int_0^1\left(x^3+\frac{x^4}{2}\right)dx$$

$$= \left[\frac{x^4}{4}+\frac{x^5}{10}\right]_0^1 = \frac{7}{20}$$

Exercise 4.5.b

$$f(x,y)=\begin{cases} 2x & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$P(X^2<Y<X) = \int_0^1\int_{x^2}^{x} 2x\,dy\,dx = \int_0^1 \left[2xy\right]_{x^2}^{x} dx$$

$$= \int_0^1\left(2x^2-2x^3\right)dx = \left[\frac{2x^3}{3}-\frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{6}$$