Solutions to exercises - Week 38

Transformations:

- Exercises 5.5 and 5.6

Estimation of the standard deviation:

- Exercises 5.11 and 5.13

F and t distributions:

- Exercises 5.17 and 5.18a-b

Order statistics:

- Exercises 5.24 and 5.25

Exercise 5.5

The pdf of $X_1 + ... + X_n$ is $f_{X_1+....+X_n}(x)$

$\bar{X} = \frac{1}{n}(X_1 + ... + X_n)$ is a scale transformation

Hence $\bar{X}$ has density (cf. section 3.5)

$$f_{\bar{X}}(x) = \frac{1}{1/n} f_{X_1+....+X_n}\left(\frac{x}{1/n}\right)$$

$$= n f_{X_1+....+X_n}(nx)$$

Exercise 5.6.b

The joint pdf of (X, Y) is $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

Consider the transformation $Z = XY$ and $W = X$

Inverse transformation: $X = W$ and $Y = Z / W$

Jacobian: $J(z,w) = \begin{vmatrix} 0 & 1 \\ 1/w & -z/w^2 \end{vmatrix} = 0 - \frac{1}{w} = -\frac{1}{w}$

Joint pdf of (Z, W):

$$f_{Z,W}(z,w) = f_{X,Y}(w,\ z/w) \cdot |-1/w| = f_X(w) f_Y(z/w) \left|\frac{1}{w}\right|$$

Marginal pdf of Z:

$$f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{w}\right| f_X(w) f_Y(z/w)\, dw$$

Exercise 5.11

The function $g(x) = x^2$ is convex

Hence by Jensen's inequality

$$\sigma^2 = \mathrm{E}S^2 = \mathrm{E}g(S) \geq g(\mathrm{E}S) = (\mathrm{E}S)^2$$

Taking square roots we obtain

$$\mathrm{E}S \leq \sqrt{\sigma^2} = \sigma$$

The inequality is strict unless $P(S^2 = a + bS) = 1$

But this is the case only if $\sigma^2 = 0$

Exercise 5.13

If $X \sim \text{gamma}(\alpha, \beta)$, then (exercise 3.17)

$$\text{E}X^v = \beta^v \frac{\Gamma(\alpha + v)}{\Gamma(\alpha)} \quad \text{for} \quad v > -\alpha$$

In particular for $U \sim \chi_p^2$ we have ($\alpha = p/2$, $\beta = 2$)

$$\text{E}U^v = 2^v \frac{\Gamma(p/2 + v)}{\Gamma(p/2)} \quad \text{for } v > -p/2$$

Now we have that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

Hence we obtain

$$\text{E}S = \frac{\sigma}{\sqrt{n-1}} \text{E}\left\{ \left(\frac{(n-1)S^2}{\sigma^2} \right)^{1/2} \right\} = \frac{\sigma}{\sqrt{n-1}} 2^{1/2} \frac{\Gamma\left(\frac{n-1}{2} + \frac{1}{2} \right)}{\Gamma\left(\frac{n-1}{2} \right)}$$

$$= \sqrt{\frac{2}{n-1}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \cdot \sigma$$

Thus

$$\hat{\sigma} = \sqrt{\frac{n-1}{2}} \frac{\Gamma((n-1)/2)}{\Gamma(n/2)} \cdot S$$

is an unbiased estimator of σ

Exercise 5.17

We have $X \sim F_{p,q}$

Then $X = (U/p)/(V/q)$, where $U \sim \chi_p^2$
and $V \sim \chi_q^2$ are independent

Question a)

The joint pdf of (U, V) is given by (for $u, v > 0$)

$$f_{U,V}(u,v) = f_U(u) f_V(v)$$

$$= \frac{1}{2^{p/2}\Gamma(p/2)} u^{p/2-1} e^{-u/2} \frac{1}{2^{q/2}\Gamma(q/2)} v^{q/2-1} e^{-v/2}$$

$$= \frac{1}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)} u^{p/2-1} v^{q/2-1} e^{-(u+v)/2}$$

We consider the transformation

$$X = (U/p)/(V/q) \qquad Y = U + V$$

The inverse transformation is

$$U = \frac{\frac{p}{q}XY}{1 + \frac{p}{q}X} \qquad V = \frac{Y}{1 + \frac{p}{q}X}$$

The Jacobian becomes

$$J(x,y) = \begin{vmatrix} \frac{(p/q)y}{[1+(p/q)x]^2} & \frac{(p/q)x}{1+(p/q)x} \\ \frac{-(p/q)y}{[1+(p/q)x]^2} & \frac{1}{1+(p/q)x} \end{vmatrix} = \frac{(p/q)y}{[1+(p/q)x]^2}$$

The joint pdf of (X, Y) is given by (for $x,y > 0$)

$$f_{X,Y}(x,y) = f_{U,V}\left(\frac{(p/q)xy}{1+(p/q)x}, \frac{y}{1+(p/q)x}\right)\frac{(p/q)y}{[1+(p/q)x]^2}$$

$$= \frac{1}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)}\left(\frac{(p/q)xy}{1+(p/q)x}\right)^{p/2-1}$$

$$\times\left(\frac{y}{1+(p/q)x}\right)^{q/2-1} e^{-y/2}\frac{(p/q)y}{[1+(p/q)x]^2}$$

$$= \frac{(p/q)^{p/2}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)}\frac{x^{p/2-1}}{(1+(p/q)x)^{(p+q)/2}} y^{(p+q)/2-1}e^{-y/2}$$

The marginal pdf of X is (for $x > 0$)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \frac{(p/q)^{p/2}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)}\frac{x^{p/2-1}}{(1+(p/q)x)^{(p+q)/2}}\int_0^{\infty} y^{(p+q)/2-1}e^{-y/2}\,dy$$

$$= \frac{(p/q)^{p/2}}{2^{(p+q)/2}\Gamma(p/2)\Gamma(q/2)}\frac{x^{p/2-1}}{(1+(p/q)x)^{(p+q)/2}} 2^{(p+q)/2}\Gamma((p+q)/2)$$

$$= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)}\left(\frac{p}{q}\right)^{p/2}\frac{x^{p/2-1}}{(1+(p/q)x)^{(p+q)/2}}$$

Question b)

$X = (U/p)/(V/q)$, where $U \sim \chi^2_p$ and $V \sim \chi^2_q$ are independent

We know that

$$\mathrm{E}U = p$$

$$\mathrm{E}U^2 = \mathrm{Var}\,U + (\mathrm{E}U)^2 = 2p + p^2$$

Further from exercise 3.17 (with $\alpha = q/2$, $\beta = 2$) we have that

$$\mathrm{E}V^m = 2^m\frac{\Gamma(q/2+m)}{\Gamma(q/2)} \quad \text{for } m > -q/2$$

By the independence of U and V, it then follows that when $q > 2$ we have

$$\mathrm{E}X = \mathrm{E}\left(\frac{U/p}{V/q}\right) = \frac{q}{p}\mathrm{E}(UV^{-1}) = \frac{q}{p}(\mathrm{E}U)(\mathrm{E}V^{-1})$$

$$= \frac{q}{p}p\,2^{-1}\frac{\Gamma(q/2-1)}{\Gamma(q/2)}$$

$$= \frac{q}{p}p\,2^{-1}\frac{\Gamma(q/2-1)}{(q/2-1)\Gamma(q/2-1)}$$

$$= \frac{q}{2}\frac{1}{(q/2-1)} = \frac{q}{q-2}$$

In a similar manner we have when $q > 4$

$$\mathrm{E}X^2 = \mathrm{E}\left(\frac{(U/p)^2}{(V/q)^2}\right) = \left(\frac{q}{p}\right)^2 (\mathrm{E}U^2)(\mathrm{E}V^{-2})$$

$$= \left(\frac{q}{p}\right)^2 (2p + p^2) 2^{-2} \frac{\Gamma(q/2-2)}{\Gamma(q/2)}$$

$$= \left(\frac{q}{p}\right)^2 (2p + p^2) 2^{-2} \frac{\Gamma(q/2-2)}{(q/2-1)(q/2-2)\Gamma(q/2-2)}$$

$$= \frac{q^2 (p+2)}{p(q-2)(q-4)}$$

Hence the variance becomes (when $q > 4$)

$$\mathrm{Var}\, X = \mathrm{E}X^2 - (\mathrm{E}X)^2 = \frac{q^2(p+2)}{p(q-2)(q-4)} - \left(\frac{q}{q-2}\right)^2$$

$$= \frac{q^2(p+2)(q-2) - q^2 p(q-4)}{p(q-2)^2(q-4)} = \frac{2q^2(p+q-2)}{p(q-2)^2(q-4)}$$

Question c)

We have that $1/X = (V/q)/(U/p)$ where $V \sim \chi^2_q$ and $U \sim \chi^2_p$ are independent

Hence $1/X \sim F_{q,p}$

Question d)

Consider the transformation

$$Y = g(X) = \frac{(p/q)X}{1+(p/q)X}$$

The inverse transformation is

$$X = g^{-1}(Y) = \frac{(q/p)Y}{1-Y}$$

Note that

$$\frac{d}{dy} g^{-1}(y) = \frac{q/p}{(1-y)^2}$$

The pdf of Y is given by (for $0 < y < 1$)

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \left(\frac{p}{q}\right)^{p/2} \frac{\left(\frac{(q/p)y}{1-y}\right)^{p/2-1}}{\left(1+(p/q)\left(\frac{(q/p)y}{1-y}\right)\right)^{(p+q)/2}} \frac{q/p}{(1-y)^2}$$

$$= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} \frac{\left(\frac{y}{1-y}\right)^{p/2-1}}{\left(1+\frac{y}{1-y}\right)^{(p+q)/2}} \frac{1}{(1-y)^2}$$

$$= \frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)} y^{p/2-1}(1-y)^{q/2-1}$$

It follows that $Y \sim \mathrm{beta}(p/2, q/2)$

Exercise 5.18

X has a Student's t distribution with $df = p$

Then $X = U/\sqrt{V/p}$, where $U \sim n(0,1)$ and $V \sim \chi^2_p$ are independent

Question a)

From exercise 3.17, we have that

$$\mathrm{E}V^m = 2^m \frac{\Gamma(p/2+m)}{\Gamma(p/2)} \quad \text{for } m > -p/2$$

By the independence of U and V, it then follows that when $p>1$ we have

$$\mathrm{E}X = \mathrm{E}\left(\frac{U}{\sqrt{V/p}}\right) = (\mathrm{E}U)\left[\mathrm{E}(p^{1/2}V^{-1/2})\right] = 0 \cdot p^{1/2}\mathrm{E}V^{-1/2} = 0$$

Further, when $p>2$ we have

$$\mathrm{Var}\, X = \mathrm{E}X^2 = \mathrm{E}\left[\left(\frac{U}{\sqrt{V/p}}\right)^2\right] = (\mathrm{E}U^2)\left[\mathrm{E}(pV^{-1})\right] = 1 \cdot p\,\mathrm{E}V^{-1}$$

$$= p\,2^{-1}\frac{\Gamma(p/2-1)}{\Gamma(p/2)} = p\,2^{-1}\frac{\Gamma(p/2-1)}{(p/2-1)\Gamma(p/2-1)}$$

$$= p\,2^{-1}\frac{1}{p/2-1} = \frac{p}{p-2}$$

Question b)

We have that

$$X^2 = \left(\frac{U}{\sqrt{V/p}}\right)^2 = \frac{U^2/1}{V/p}$$

Now $U^2 \sim \chi^2_1$ and $V \sim \chi^2_p$ are independent, and it follows that $X^2 \sim F_{1,p}$

Exercise 5.24

The joint pdf of the i-th and j-th order statistic is given by ($1 \le i < j \le n$)

$$f_{X_{(i)},X_{(j)}}(u,v) = \frac{n!}{(i-1)!\,(j-1-i)!\,(n-j)!} \times f_X(u)\,f_X(v)\left[F_X(u)\right]^{i-1}\left[F_X(v)-F_X(u)\right]^{j-1-i}\left[1-F_X(v)\right]^{n-j}$$

For $i = 1$ and $j = n$ this gives

$$f_{X_{(1)},X_{(n)}}(u,v) = \frac{n!}{(n-2)!} f_X(u)\,f_X(v)\left[F_X(v)-F_X(u)\right]^{n-2}$$

$$= n(n-1)\, f_X(u)\,f_X(v)\left[F_X(v)-F_X(u)\right]^{n-2}$$

Here we have

$$f_X(x) = 1/\theta \quad \text{if } 0 < x < \theta$$

$$F_X(x) = x/\theta \quad \text{if } 0 < x < \theta$$

Therefore (for $0 < u < v < \theta$)

$$f_{X_{(1)},X_{(n)}}(u,v) = n(n-1)\, f_X(u)\, f_X(v)\left[F_X(v) - F_X(u)\right]^{n-2}$$

$$= n(n-1)\frac{1}{\theta}\frac{1}{\theta}\left[\frac{v}{\theta} - \frac{u}{\theta}\right]^{n-2}$$

$$= n(n-1)\frac{(v-u)^{n-2}}{\theta^n}$$

Consider the transformation $Z = X_{(1)} / X_{(n)}$ and $W = X_{(n)}$

Inverse transformation: $X_{(1)} = ZW$ and $X_{(n)} = W$

Jacobian: $J(z,w) = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$

The joint pdf of (Z, W) is given by
(for $0 < z < 1, \quad 0 < w < \theta$)

$$f_{Z,W}(z,w) = f_{X_{(1)},X_{(n)}}(zw, w)\,|w| = n(n-1)\frac{(w - zw)^{n-2}}{\theta^n} w$$

$$= \frac{n(n-1)}{\theta^n} w^{n-1}(1-z)^{n-2}$$

The joint pdf may be factorized, so Z and W are independent

Exercise 5.25

The joint pdf of all the order statistic is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) = \begin{cases} n!\, f_X(x_1)\cdot\ldots\cdot f_X(x_n) & \text{if } x_1 < \ldots < x_n \\ 0 & \text{otherwise} \end{cases}$$

Here we have

$$f_X(x) = \frac{a}{\theta^a} x^{a-1} \quad \text{if } 0 < x < \theta$$

Therefore (for $0 < x_1 < \ldots < x_n < \theta$)

$$f_{X_{(1)},\ldots,X_{(n)}}(x_1,\ldots,x_n) = \frac{n!\, a^n}{\theta^{na}} x_1^{a-1}\cdot\ldots\cdot x_n^{a-1}$$

Here we consider the transformation

$$Y_1 = \frac{X_{(1)}}{X_{(2)}},\ Y_2 = \frac{X_{(2)}}{X_{(3)}},\ \ldots,\ Y_{n-1} = \frac{X_{(n-1)}}{X_{(n)}},\ Y_n = X_{(n)}$$

The inverse transformation is

$$X_{(1)} = Y_1\cdot\ldots\cdot Y_n,\quad X_{(2)} = Y_2\cdot\ldots\cdot Y_n,\quad \ldots\quad X_{(n-1)} = Y_{n-1}Y_n,\quad X_{(n)} = Y_n$$

The Jacobian becomes

$$J(y_1,\ldots,y_n) = \begin{vmatrix} \prod_{i\geq 2} y_i & \prod_{i\neq 2} y_i & \prod_{i\neq 3} y_i & \cdots & \prod_{i\neq n} y_i \\ 0 & \prod_{i\geq 3} y_i & \prod_{i\geq 2,\, i\neq 3} y_i & \cdots & \prod_{i\geq 2,\, i\neq n} y_i \\ 0 & 0 & \prod_{i\geq 4} y_i & \cdots & \prod_{i\geq 3,\, i\neq n} y_i \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = y_2 y_3^2\cdot\ldots\cdot y_n^{n-1}$$

The joint pdf of $(Y_1, \ldots, Y_n)$ is given by
(for $0 < y_i < 1;\ i = 1, \ldots, n-1;\quad 0 < y_n < \theta$)

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = f_{X_{(1)},\ldots,X_{(n)}}(y_1 \cdot \ldots \cdot y_n, y_2 \cdot \ldots \cdot y_n, \ldots, y_n)\left|J\right|$$

$$= \frac{n!\,a^n}{\theta^{na}}\left(y_1 \cdot \ldots \cdot y_n\right)^{a-1}\left(y_2 \cdot \ldots \cdot y_n\right)^{a-1} \cdot \ldots \cdot y_n^{a-1} y_2 y_3^2 \cdot \ldots \cdot y_n^{n-1}$$

$$= \frac{n!\,a^n}{\theta^{na}} y_1^{a-1} y_2^{2a-1} \cdot \ldots \cdot y_n^{na-1} = \left(\prod_{i=1}^{n-1} ia y_i^{ia-1}\right)\frac{na}{\theta^{na}} y_n^{na-1}$$

We see that $Y_1, \ldots, Y_n$ are independent and that

$$f_{Y_i}(y_i) = ia y_i^{ia-1}\ ;\ i = 1, \ldots, n-1$$

and

$$f_{Y_n}(y_n) = \frac{na}{\theta^{na}} y_n^{na-1}$$