Solutions to exercises - Week 40

Moment and maximum likelihood estimators:

- Exercises 7.9, 7.11 and 7.13

Bayes estimators:

- Exercises 7.22 and 7.24

Best unbiased estimators:

- Exercise 7.40 (solution is given in the lectures for week 40)

Exercise 7.9

$X_1, \ldots, X_n$ are iid with pdf

$$f(x|\theta) = 1/\theta \qquad \text{for } 0 \le x \le \theta$$

We have that $\text{E}X = \theta/2$ and $\text{Var}X = \theta^2/12$

The moment estimator is given by the equation $\bar{X} = \theta/2$ so it is given by $\theta^* = 2\bar{X}$

We have

$$\text{E}\theta^* = 2\text{E}\bar{X} = 2(\theta/2) = \theta$$

$$\text{Var}\theta^* = 4\text{Var}\bar{X} = 4\frac{1}{n}\left(\frac{\theta^2}{12}\right) = \frac{\theta^2}{3n}$$

Since θ^* is unbiased, its MSE equals the variance

The likelihood is given by

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}(x_i)$$

$$= \frac{1}{\theta^n}\prod_{i=1}^{n} I_{[0,\theta]}(x_i) = \frac{1}{\theta^n} I_{[0,\infty)}(\min x_i)\, I_{(-\infty,\theta]}(\max x_i)$$

We see that the likelihood is zero for $\theta < \max x_i$ and that it is decreasing in θ for $\theta \ge \max x_i$

Thus the maximum value of the likelihood is obtained for $\theta = \max x_i$

It follows that the ML-estimator is

$$\hat{\theta} = X_{(n)} = \max X_i$$

We know that the j-th order statistic is has pdf

$$f_{X_{(j)}}(x|\theta) = \frac{n!}{(j-1)!\,(n-j)!} f(x|\theta)\left[F(x|\theta)\right]^{j-1}\left[1-F(x|\theta)\right]^{n-j}$$

Hence $X_{(n)} = \max X_i$ has pdf (for $0 \le x \le \theta$)

$$f_{X_{(n)}}(x|\theta) = n\, f(x|\theta)\left[F(x|\theta)\right]^{n-1} = n\frac{1}{\theta}\left(\frac{x}{\theta}\right)^{n-1} = \frac{n}{\theta^n}x^{n-1}$$

This gives

$$E\hat{\theta}^k = \int_{-\infty}^{\infty} x^k f_{X_{(n)}}(x|\theta)\,dx = \int_0^{\theta} x^k \frac{n}{\theta^n}x^{n-1}\,dx$$

$$= \frac{n}{\theta^n}\int_0^{\theta} x^{n+k-1}\,dx = \frac{n}{\theta^n}\left[\frac{x^{n+k}}{n+k}\right]_0^{\theta} = \frac{n}{n+k}\theta^k$$

Hence we have

$$\mathrm{E}\hat{\theta} = \frac{n}{n+1}\theta$$

$$\mathrm{Var}\hat{\theta} = \mathrm{E}\hat{\theta}^2 - (\mathrm{E}\hat{\theta})^2 = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$$

Thus the mean squared error of the MLE is

$$\mathrm{E}(\hat{\theta}-\theta)^2 = \mathrm{Var}\,\hat{\theta} + \left(\mathrm{Bias}\,\hat{\theta}\right)^2 = \frac{n\theta^2}{(n+2)(n+1)^2} + \left(\frac{n}{n+1}\theta - \theta\right)^2$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2} + \frac{\theta^2}{(n+1)^2} = \frac{2\theta^2}{(n+2)(n+1)}$$

We find that the ML-estimator has smaller MSE than the moment estimator when $n \geq 3$

Exercise 7.11

$X_1, \ldots, X_n$ are iid with pdf

$$f(x\,|\,\theta) = \theta x^{\theta-1} \qquad \text{for } 0 \leq x \leq 1$$

a) The likelihood is given by (when $0 \leq x_i \leq 1$ all i)

$$L(\theta\,|\,\mathbf{x}) = \prod_{i=1}^{n} f(x_i\,|\,\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

The log-likelihood becomes

$$\log L(\theta\,|\,\mathbf{x}) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

Hence we have

$$\frac{\partial}{\partial\theta}\log L(\theta\,|\,\mathbf{x}) = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i$$

We solve $\partial \log L(\theta\,|\,\mathbf{x}) / \partial\theta = 0$ and find that the MLE is given by

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}\left(-\log X_i\right)}$$

In order to find the variance of $\hat{\theta}$, we first find the distribution of $Y_i = g(X_i) = -\log X_i$

The inverse transformation is $X_i = g^{-1}(Y_i) = e^{-Y_i}$

Hence Y_i has density (for $y > 0$)

$$f_Y(y\,|\,\theta) = f(e^{-y}\,|\,\theta)e^{-y} = \theta\left(e^{-y}\right)^{\theta-1} e^{-y} = \theta e^{-\theta y}$$

Thus $Y_i \sim \mathrm{gamma}(1, 1/\theta)$

It then follows that

$$T = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n}\left(-\log X_i\right) \sim \mathrm{gamma}(n, 1/\theta)$$

Now the MLE is given by

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}\left(-\log X_i\right)} = \frac{n}{T}$$

Now from exercise 3.17 (with $\alpha = n$, $\beta = 1/\theta$) we have that

$$\mathrm{E}T^k = \left(\frac{1}{\theta}\right)^k \frac{\Gamma(n+k)}{\Gamma(n)} \qquad \text{for } k > -n$$

Thus we have

$$E\hat{\theta} = E\left(\frac{n}{T}\right) = n\left(\frac{1}{\theta}\right)^{-1}\frac{\Gamma(n-1)}{\Gamma(n)} = n\theta\frac{\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \frac{n}{n-1}\theta$$

$$E\hat{\theta}^2 = E\left(\frac{n^2}{T^2}\right) = n^2\left(\frac{1}{\theta}\right)^{-2}\frac{\Gamma(n-2)}{\Gamma(n)} = \frac{n^2\theta^2}{(n-1)(n-2)}$$

$$\text{Var}\hat{\theta} = E\hat{\theta}^2 - (E\hat{\theta}) = \frac{n^2\theta^2}{(n-1)(n-2)} - \left(\frac{n}{n-1}\theta\right)^2 = \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

We see that the variance goes to zero as n goes to infinity

b) We have that $X_1,...,X_n$ are iid with pdf

$$f(x|\theta) = \theta x^{\theta-1} \quad \text{for } 0 \le x \le 1$$

We have that $X_i \sim \text{beta}(\theta,1)$

Then $EX_i = \theta/(\theta+1)$ and the moment estimator is given by

$$\frac{\theta^*}{\theta^*+1} = \bar{X}$$

which gives

$$\theta^* = \frac{\bar{X}}{1-\bar{X}}$$

Exercise 7.13

$X_1,...,X_n$ are iid with pdf $f(x|\theta) = \frac{1}{2}e^{-|x-\theta|}$

The likelihood is given by

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n}\frac{1}{2}e^{-|x_i-\theta|} = \frac{1}{2^n}e^{-\sum_{i=1}^{n}|x_i-\theta|}$$

Thus the log-likelihood may be written

$$\log L(\theta|\mathbf{x}) = -n\log 2 - \sum_{i=1}^{n}|x_i-\theta|$$
$$= -n\log 2 - \sum_{i=1}^{n}|x_{(i)}-\theta|$$

where $x_{(1)} < ... < x_{(n)}$ are the x_i written in increasing order

Note that the log-likelihood is continuous for all θ and differentiable except when $\theta \in \{x_{(1)},....,x_{(n)}\}$

In order to find the derivative of the log-likelihood, we note that we for $x_{(j)} < \theta < x_{(j+1)}$ may write

$$\log L(\theta|\mathbf{x}) = -n\log 2 - \sum_{i=1}^{j}(\theta - x_{(i)}) - \sum_{i=j+1}^{n}(x_{(i)} - \theta)$$
$$= -n\log 2 - j\theta + \sum_{i=1}^{j}x_{(i)} - \sum_{i=j+1}^{n}x_{(i)} + (n-j)\theta$$
$$= -n\log 2 + (n-2j)\theta + \sum_{i=1}^{j}x_{(i)} - \sum_{i=j+1}^{n}x_{(i)}$$

Thus for $x_{(j)} < \theta < x_{(j+1)}$ we have

$$\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) = n-2j$$

It follows that the log-likelihood is continuous and piecewise linear, and that its slope is

- positive if $j < n/2$
- equal to zero if $j = n/2$
- negative if $j > n/2$

If n is odd, the log-likelihood is maximized for $\theta = x_{((n+1)/2)}$

If n is even, the log-likelihood is maximized for
$\theta \in \left[x_{(n/2)},\, x_{(n/2+1)}\right]$

Thus the MLE is the sample median

$$M = \begin{cases} X_{((n+1)/2)} & \text{if } n \text{ is odd} \\ \frac{1}{2}\left(X_{(n/2)} + X_{(n/2+1)}\right) & \text{if } n \text{ is even} \end{cases}$$

Exercise 7.22

Given θ we have $X_1, X_2, \ldots, X_n$ iid $n(\theta, \sigma^2)$

Further $\theta \sim n(\mu, \tau^2)$

a) Given θ we have $\overline{X} \sim n(\theta, \sigma^2/n)$, so

$$f(\overline{x} \mid \theta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n(\overline{x}-\theta)^2/(2\sigma^2)}$$

The prior distribution is

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}\tau} e^{-(\theta-\mu)^2/(2\tau^2)}$$

Thus the joint distribution of $\overline{X}$ and θ is

$$f(\overline{x}, \theta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-n(\overline{x}-\theta)^2/(2\sigma^2)} \frac{1}{\sqrt{2\pi}\tau} e^{-(\theta-\mu)^2/(2\tau^2)}$$

b) The marginal pdf of $\overline{X}$ is given by

$$m(\overline{x}) = \int f(\overline{x}, \theta) d\theta = \int_{-\infty}^{\infty} \frac{\sqrt{n}}{2\pi\sigma\tau} e^{-K/2} \, d\theta$$

Here we have

$$K = \frac{n}{\sigma^2}(\overline{x}-\theta)^2 + \frac{1}{\tau^2}(\theta-\mu)^2$$

$$= \frac{n}{\sigma^2}\left(\overline{x}^2 - 2\overline{x}\theta + \theta^2\right) + \frac{1}{\tau^2}\left(\theta^2 - 2\mu\theta + \mu^2\right)$$

$$= \left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)\theta^2 - 2\left(\frac{n}{\sigma^2}\overline{x} + \frac{1}{\tau^2}\mu\right)\theta + \frac{n}{\sigma^2}\overline{x}^2 + \frac{1}{\tau^2}\mu^2$$

$$= \frac{\tau^2 + \sigma^2/n}{\tau^2\sigma^2/n}\left\{\theta^2 - 2\left(\frac{\overline{x}\tau^2 + \mu\sigma^2/n}{\tau^2 + \sigma^2/n}\right)\theta\right\} + \frac{n}{\sigma^2}\overline{x}^2 + \frac{1}{\tau^2}\mu^2$$

$$= \frac{\tau^2 + \sigma^2/n}{\tau^2\sigma^2/n}\left(\theta - \frac{\overline{x}\tau^2 + \mu\sigma^2/n}{\tau^2 + \sigma^2/n}\right)^2 - \frac{\left(\overline{x}\tau^2 + \mu\sigma^2/n\right)^2}{\left(\tau^2 + \sigma^2/n\right)\tau^2\sigma^2/n} + \frac{n}{\sigma^2}\overline{x}^2 + \frac{1}{\tau^2}\mu^2$$

$$= \frac{1}{\frac{\tau^2\sigma^2/n}{\tau^2 + \sigma^2/n}}\left(\theta - \frac{\overline{x}\tau^2 + \mu\sigma^2/n}{\tau^2 + \sigma^2/n}\right)^2 + \frac{\overline{x}^2 - 2\overline{x}\mu + \mu^2}{\tau^2 + \sigma^2/n}$$

$$= \frac{1}{\omega^2}\left(\theta - \eta(\overline{x})\right)^2 + \frac{1}{\tau^2 + \sigma^2/n}\left(\overline{x} - \mu\right)^2$$

where

$$\eta(\overline{x}) = \frac{\overline{x}\tau^2 + \mu\sigma^2/n}{\tau^2 + \sigma^2/n} \quad \text{and} \quad \omega^2 = \frac{\tau^2\sigma^2/n}{\tau^2 + \sigma^2/n}$$

Thus the marginal distribution of $\bar{X}$ is given by

$$m(\bar{x}) = \int_{-\infty}^{\infty} \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2\omega^2}(\theta-\eta(\bar{x}))^2 - \frac{1}{2(\tau^2+\sigma^2/n)}(\bar{x}-\mu)^2\right\} d\theta$$

$$= \frac{\omega}{\sqrt{2\pi}\,(\sigma\tau/\sqrt{n})} \exp\left\{-\frac{1}{2(\tau^2+\sigma^2/n)}(\bar{x}-\mu)^2\right\} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\omega} \exp\left\{-\frac{1}{2\omega^2}(\theta-\eta(\bar{x}))^2\right\} d\theta$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\tau^2+\sigma^2/n}} \exp\left\{-\frac{1}{2(\tau^2+\sigma^2/n)}(\bar{x}-\mu)^2\right\}$$

This shows that

$$\bar{X} \sim n\left(\mu, \tau^2 + \sigma^2/n\right)$$

c) Finally we find that the posterior distribution of θ given $\bar{X} = \bar{x}$ is given by

$$\pi(\theta \mid \bar{x}) = \frac{\frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left\{-\frac{1}{2\omega^2}(\theta-\eta(\bar{x}))^2 - \frac{1}{2(\tau^2+\sigma^2/n)}(\bar{x}-\mu)^2\right\}}{\frac{1}{\sqrt{2\pi}\sqrt{\tau^2+\sigma^2/n}} \exp\left\{-\frac{1}{2(\tau^2+\sigma^2/n)}(\bar{x}-\mu)^2\right\}}$$

$$= \frac{\sqrt{\tau^2+\sigma^2/n}}{\sqrt{2\pi}\,(\sigma\tau/\sqrt{n})} \exp\left\{-\frac{1}{2\omega^2}(\theta-\eta(\bar{x}))^2\right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\omega} \exp\left\{-\frac{1}{2\omega^2}(\theta-\eta(\bar{x}))^2\right\}$$

So

$$\theta \mid \bar{X} = \bar{x} \sim n(\eta(\bar{x}), \omega^2)$$

Exercise 7.24

Given λ we have $X_1, X_2, \ldots, X_n$ iid Poisson(λ)

Further $\lambda \sim \text{gamma}(\alpha, \beta)$

a) We have

$$f(x_1,\ldots,x_n \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) \lambda^{\sum_{i=1}^{n} x_i} e^{-\lambda}$$

and

$$\pi(\lambda) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

Thus the joint distribution of $X_1, \ldots, X_n$ and λ is

$$f(x_1,\ldots,x_n,\lambda) = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) \lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda} \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda/\beta}$$

If we let $y = \sum_{i=1}^{n} x_i$ we may write

$$f(x_1,\ldots,x_n,\lambda) = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{y+\alpha-1} e^{-n\lambda-\lambda/\beta}$$

$$= \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{y+\alpha-1} e^{-\frac{\lambda}{\beta/(n\beta+1)}}$$

The marginal distribution of $X_1, X_2, \ldots, X_n$ becomes

$$m(x_1,\ldots,x_n) = \int f(x_1,\ldots,x_n,\lambda)\, d\lambda$$

$$= \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \lambda^{y+\alpha-1} e^{-\frac{\lambda}{\beta/(n\beta+1)}} d\lambda$$

$$= \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{n\beta+1}\right)^{y+\alpha} \Gamma(y+\alpha)$$

Thus the posterior distribution of λ is given by

$$f(\lambda \mid x_1,\dots,x_n) = \frac{f(x_1,\dots,x_n,\lambda)}{m(x_1,\dots,x_n)}$$

$$= \frac{\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\lambda^{y+\alpha-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}}{\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}\Gamma(y+\alpha)}$$

$$= \frac{1}{\left(\frac{\beta}{n\beta+1}\right)^{y+\alpha}\Gamma(y+\alpha)}\lambda^{y+\alpha-1}e^{-\frac{\lambda}{\beta/(n\beta+1)}}$$

Thus

$$\lambda \mid X_1 = x_1,\dots,X_n = x_n \sim \text{gamma}(y+\alpha\ ,\ \beta/(n\beta+1))$$

b) The posterior mean is:

$$\mathrm{E}(\lambda \mid \sum_{i=1}^{n} X_i = y) = (y+\alpha)\frac{\beta}{n\beta+1}$$

$$= \frac{n\beta}{n\beta+1}\frac{y}{n} + \frac{1}{n\beta+1}(\alpha\beta)$$

The posterior variance is:

$$\mathrm{Var}(\lambda \mid \sum_{i=1}^{n} X_i = y) = (y+\alpha)\left(\frac{\beta}{n\beta+1}\right)^2$$