Solutions to exercises - Week 41

Maximum likelihood estimators:

- Exercise 7.19

Cramér-Rao inequality:

- Exercise 7.38

Best linear unbiased estimators:

- Exercises 7.41 and 7.42

Exercise 7.19

$Y_1,...,Y_n$ are given by $Y_i = \beta x_i + \varepsilon_i$, where $x_1,...,x_n$ are fixed constants and $\varepsilon_1,...,\varepsilon_n$ are iid $n(0,\sigma^2)$

a) Note that $Y_i \sim n(\beta x_i, \sigma^2)$ so the joint pdf of $Y_1,...,Y_n$ is given by

$$f(y_1,...,y_n \mid \beta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2\right\}$$

$$= \left(2\pi\sigma^2\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(y_i^2 - 2\beta x_i y_i + \beta^2 x_i^2\right)\right\}$$

$$= \left(2\pi\sigma^2\right)^{-n/2} \exp\left\{-\frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right\} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i\right\}$$

By the factorization theorem we have that

$\left(\sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} x_i Y_i\right)$ is sufficient

b) The likelihood is given by

$$L(\beta,\sigma^2 \mid \mathbf{y}) = f(y_1,...,y_n \mid \beta,\sigma^2)$$

$$= \left(2\pi\sigma^2\right)^{-n/2} \exp\left\{-\frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2\right\} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i\right\}$$

The log-likelihood becomes

$$\log L(\beta,\sigma^2 \mid \mathbf{y}) = -\frac{n}{2}\log\left(2\pi\sigma^2\right) - \frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i$$

Now

$$\frac{\partial}{\partial\beta}\log L(\beta,\sigma^2 \mid \mathbf{y}) = -\frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i y_i$$

If we set the derivative equal to zero, we find that for any value of σ^2 the value of β that maximizes the log-likelihood is given by

$$\hat{\beta} = \sum_{i=1}^{n} x_i y_i \Big/ \sum_{i=1}^{n} x_i^2$$

Thus the MLE of β is given by

$$\hat{\beta} = \sum_{i=1}^{n} x_i Y_i \Big/ \sum_{i=1}^{n} x_i^2$$

We find

$$\mathrm{E}\hat{\beta} = \sum_{i=1}^{n} x_i \mathrm{E}Y_i \Big/ \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i(\beta x_i) \Big/ \sum_{i=1}^{n} x_i^2 = \beta$$

c) $\hat{\beta}$ is a linear combination of normally distributed random variables, so it is itself normally distributed

The variance becomes

$$\mathrm{Var}\hat{\beta} = \frac{\sum_{i=1}^{n} x_i^2 \ \mathrm{Var}Y_i}{\left(\sum_{i=1}^{n} x_i^2\right)^2} = \frac{\sigma^2 \sum_{i=1}^{n} x_i^2}{\left(\sum_{i=1}^{n} x_i^2\right)^2} = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$

Exercise 7.38.a

$X_1,...,X_n$ are iid with pdf $f(x|\theta)=\theta x^{\theta-1} \quad (0<x<1)$

The joint pdf of $\mathbf{X}=(X_1,...,X_n)$ is (when $0<x_i<1$)

$$f(\mathbf{x}|\theta)=\prod_{i=1}^{n} f(x_i|\theta)=\prod_{i=1}^{n}\theta x_i^{\theta-1}=\theta^n\prod_{i=1}^{n}x_i^{\theta-1}$$

Hence we have

$$\log f(\mathbf{X}|\theta)=n\log\theta+(\theta-1)\sum_{i=1}^{n}\log X_i$$

It follows that

$$\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta)=\frac{n}{\theta}+\sum_{i=1}^{n}\log X_i \;=-n\left(-\frac{1}{n}\sum_{i=1}^{n}\log X_i-\frac{1}{\theta}\right)$$

Thus $-(1/n)\sum_{i=1}^{n}\log X_i$ is the UMVUE for $1/\theta$ and it attains the Cramér-Rao lower bound

Exercise 7.38.b

$X_1,...,X_n$ are iid with pdf

$$f(x|\theta)=\frac{\log\theta}{\theta-1}\theta^x \qquad \text{for } \; 0<x<1,\; \theta>1$$

The joint pdf of $\mathbf{X}=(X_1,...,X_n)$ is (when $0<x_i<1$)

$$f(\mathbf{x}|\theta)=\prod_{i=1}^{n} f(x_i|\theta)=\prod_{i=1}^{n}\frac{\log\theta}{\theta-1}\theta^{x_i}=\left(\frac{\log\theta}{\theta-1}\right)^n\prod_{i=1}^{n}\theta^{x_i}$$

Hence we have

$$\log f(\mathbf{X}|\theta)=n\log(\log\theta)-n\log(\theta-1)+\log\theta\sum_{i=1}^{n}X_i$$

It follows that

$$\frac{\partial}{\partial\theta}\log f(\mathbf{X}|\theta)=n\frac{1/\theta}{\log\theta}-\frac{n}{\theta-1}+\frac{1}{\theta}\sum_{i=1}^{n}X_i$$

$$=\frac{n}{\theta}\left(\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{\theta}{\theta-1}+\frac{1}{\log\theta}\right)$$

$$=\frac{n}{\theta}\left\{\bar{X}-\left(\frac{\theta}{\theta-1}-\frac{1}{\log\theta}\right)\right\}$$

Thus $\bar{X}$ is the UMVUE for $\frac{\theta}{\theta-1}-\frac{1}{\log\theta}$ and it attains the Cramér-Rao lower bound

Exercise 7.41

$X_1,...,X_n$ is a random sample from a population with mean μ and variance σ^2

a) The estimator $W=\sum_{i=1}^{n}a_iX_i$ has mean

$$\mathrm{E}W=\sum_{i=1}^{n}a_i\mathrm{E}X_i=\mu\sum_{i=1}^{n}a_i$$

so it is unbiased if $\sum_{i=1}^{n}a_i=1$

b) Before we solve the problem, we recall that the Cauchy-Schwartz inequality states that

$$\left(\sum_{i=1}^{n}x_iy_i\right)^2\le\left(\sum_{i=1}^{n}x_i^2\right)\left(\sum_{i=1}^{n}y_i^2\right)$$

with equality if and only if $y_i=kx_i$

Now the variance of W becomes

$$\mathrm{Var}W = \sum_{i=1}^{n} a_i^2 \mathrm{Var}X_i = \sigma^2 \sum_{i=1}^{n} a_i^2$$

By the Cauchy-Schwartz inequality we now obtain (set $x_i = a_i$ and $y_i = 1$)

$$\sum_{i=1}^{n} a_i^2 \geq \frac{\left(\sum_{i=1}^{n} a_i \cdot 1\right)^2}{\sum_{i=1}^{n} 1^2} = \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{n}$$

with equality if and only if $a_i = k \cdot 1$

If we combine this with the result in a), we see that the variance is minimized for $a_i = 1/n$, i.e. when $W = \bar{X}$

Exercise 7.42

$W_1, ..., W_k$ are unbiased estimators of a parameter θ with $\mathrm{Var}W_i = \sigma_i^2$ and $\mathrm{Cov}(W_i, W_j) = 0$ for $i \neq j$

We consider estimators of the form

$$\hat{W} = \sum_{i=1}^{k} a_i W_i$$

and assume that they are unbiased

This implies that $\sum_{i=1}^{k} a_i = 1$ (cf. exercise 7.41.a)

The variance is given by

$$\mathrm{Var}\hat{W} = \sum_{i=1}^{k} a_i^2 \mathrm{Var}W_i = \sum_{i=1}^{k} a_i^2 \sigma_i^2$$

By the Cauchy-Schwartz inequality we now obtain (set $x_i = a_i \sigma_i$ and $y_i = 1/\sigma_i$)

$$\sum_{i=1}^{k} a_i^2 \sigma_i^2 \geq \frac{\left(\sum_{i=1}^{k} a_i \sigma_i \cdot (1/\sigma_i)\right)^2}{\sum_{i=1}^{k} (1/\sigma_i)^2} = \frac{\left(\sum_{i=1}^{k} a_i\right)^2}{\sum_{i=1}^{k} 1/\sigma_i^2}$$

with equality if and only if

$$a_i \sigma_i = C \cdot (1/\sigma_i)$$

i.e. if and only if

$$a_i = C / \sigma_i^2$$

Thus the estimator with minimum variance is

$$W^* = \frac{\sum_{i=1}^{k} W_i / \sigma_i^2}{\sum_{j=1}^{k} 1/\sigma_j^2}$$

Its variance becomes

$$\mathrm{Var}W^* = \frac{\sum_{i=1}^{k} \left(1/\sigma_i^2\right)^2 \mathrm{Var}W_i}{\left(\sum_{j=1}^{k} 1/\sigma_j^2\right)^2} = \frac{\sum_{i=1}^{k} 1/\sigma_i^2}{\left(\sum_{j=1}^{k} 1/\sigma_j^2\right)^2} = \frac{1}{\sum_{j=1}^{k} 1/\sigma_j^2}$$