Solutions to exercises - Week 43

Convergence to infinity in probability:

- Exercise 5.33

Central limit theorem:

- Exercise 5.35

Relation between convergence in probability and convergence in distribution:

- Exercise 5.41

Convergence in distribution:

- Exercise 5.42

Delta method:

- Exercise 5.44

Exercise 5.33

Let $X_1, X_2, \ldots.$ be a sequence of random variables that converges in distribution to X, and let $Y_1, Y_2, \ldots.$ be a sequence of random variables that converges in probability to infinity, i.e. for any $c > 0$ we have that $\lim_{n\to\infty} P(Y_n > c) = 1$

We want to prove that $X_1 + Y_1, X_2 + Y_2, \ldots.$ converges in probability to infinity

To this end we have to prove that for any $C > 0$ and $\varepsilon > 0$ there exists an N such that for all $n \geq N$ we have

$$P(X_n + Y_n > C) > 1 - \varepsilon$$

There exist a $k > 0$, which is a point of continuity of $F_X(x)$, such that

$$P(X \leq -k) = F_X(-k) \leq \varepsilon / 3$$

Now $P(X_n \leq -k) \to P(X \leq -k)$

so there exists an N_1 such that

$$|P(X_n \leq -k) - P(X \leq -k)| < \varepsilon / 3$$

for $n \geq N_1$

Hence for $n \geq N_1$ we have

$$P(X_n \leq -k) \leq P(X \leq -k) + |P(X_n \leq -k) - P(X \leq -k)|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$$

Further, since $Y_1, Y_2, \ldots.$ converges to infinity in probability, there exists an N_2 such that $P(Y_n > C + k) > 1 - \varepsilon / 3$ for $n \geq N_2$

Then we have for $n \geq N = \max(N_1, N_2)$

$$P(X_n + Y_n > C) \geq P((X_n > -k) \cap (Y_n > C + k))$$

$$= 1 - P\left((X_n > -k)^c \cup (Y_n > C + k)^c\right)$$

$$\geq 1 - P\left((X_n > -k)^c\right) - P\left((Y_n > C + k)^c\right)$$

$$= 1 - P(X_n \leq -k) - [1 - P(Y_n > C + k)]$$

$$= P(Y_n > C + k) - P(X_n \leq -k)$$

$$> 1 - \frac{\varepsilon}{3} - \frac{2\varepsilon}{3} = 1 - \varepsilon$$

which proves the result

Exercise 5.35

Let $X_1, X_2, ...$ be iid and exponential(1)

a) We have $\mathrm{E}X_i = 1$ and $\mathrm{Var}X_i = 1$

By the central limit theorem (CLT) we then have

$$\sqrt{n}\left(\bar{X}_n - 1\right) \rightarrow n(0,1) \text{ in distribution}$$

Thus we have

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \le x\right) \rightarrow P(Z \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

b) Now we have that

$$\frac{d}{dx} P(Z \le x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Further $W = \sum_{i=1}^{n} X_i \sim \mathrm{gamma}(n,1)$

Hence we obtain

$$\frac{d}{dx} P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \le x\right) = \frac{d}{dx} P\left(\bar{X}_n \le \frac{x}{\sqrt{n}} + 1\right) = \frac{d}{dx} P\left(\sum_{i=1}^{n} X_i \le x\sqrt{n} + n\right)$$

$$= \frac{d}{dx} F_W\left(x\sqrt{n} + n\right) = f_W\left(x\sqrt{n} + n\right)\sqrt{n}$$

$$= \frac{1}{\Gamma(n)} (x\sqrt{n} + n)^{n-1} e^{-(x\sqrt{n}+n)} \sqrt{n}$$

From the approximation

$$\frac{d}{dx} P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} \le x\right) \approx \frac{d}{dx} P(Z \le x)$$

we obtain

$$\frac{1}{\Gamma(n)} (x\sqrt{n} + n)^{n-1} e^{-(x\sqrt{n}+n)} \sqrt{n} \approx \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Substituting $x = 0$ we get

$$\frac{\sqrt{n}}{(n-1)!} n^{n-1} e^{-n} \approx \frac{1}{\sqrt{2\pi}}$$

From this we obtain Stirling's formula:

$$n! = (n-1)!\,n$$

$$\approx (\sqrt{2\pi n}) \cdot n \cdot n^{n-1} e^{-n} = \sqrt{2\pi}\, n^{n+(1/2)}\, e^{-n}$$

Exercise 5.41

Assume that

$$P(|X_n - \mu| > \varepsilon) \rightarrow 0 \quad \text{for every } \varepsilon > 0$$

Then we have for $x > \mu$

$$P(X_n \le x) = 1 - P(X_n > x) = 1 - P(X_n - \mu > x - \mu)$$

$$\ge 1 - P(|X_n - \mu| > x - \mu) \rightarrow 1$$

Similarly for $x < \mu$

$$P(X_n \le x) = P(\mu - X_n \ge \mu - x)$$

$$\le P(|X_n - \mu| \ge \mu - x) \rightarrow 0$$

Thus we have proved that

$$P(X_n \le x) \rightarrow \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x > \mu \end{cases}$$

Then we assume that

$$P(X_n \le x) \to \begin{cases} 0 & \text{if } x < \mu \\ 1 & \text{if } x > \mu \end{cases}$$

Let $\varepsilon > 0$. Then

$$P(|X_n - \mu| > \varepsilon) = P(X_n - \mu < -\varepsilon) + P(X_n - \mu > \varepsilon)$$

$$\le P(X_n \le \mu - \varepsilon) + P(X_n > \mu + \varepsilon)$$

$$\le P(X_n \le \mu - \varepsilon) + [1 - P(X_n \le \mu + \varepsilon)]$$

$$\to 0 + (1-1) = 0$$

which proves the result

Exercise 5.42.a

Let $X_1, X_2, ...$ be iid $\text{beta}(1, \beta)$

We have (for $0 < x < 1$)

$$f_X(x) = \beta(1-x)^{\beta-1}$$

and

$$F_X(x) = \int_0^x \beta(1-u)^{\beta-1} du$$

$$= \left[-(1-u)^{\beta}\right]_0^x$$

$$= 1 - (1-x)^{\beta}$$

Consider $X_{(n)} = \max_{1 \le i \le n} X_i$

Then we have

$$P\left(n^{\nu}(1 - X_{(n)}) \le x\right) = 1 - P\left(n^{\nu}(1 - X_{(n)}) > x\right)$$

$$= 1 - P\left(X_{(n)} < 1 - \frac{x}{n^{\nu}}\right) = 1 - P\left(X_i < 1 - \frac{x}{n^{\nu}} \text{ for } i = 1, ..., n\right)$$

$$= 1 - \prod_{i=1}^{n} P\left(X_i < 1 - \frac{x}{n^{\nu}}\right) \quad = 1 - \left[F_X\left(1 - x/n^{\nu}\right)\right]^n$$

$$= 1 - \left[1 - \left(x/n^{\nu}\right)^{\beta}\right]^n \quad = 1 - \left[1 - \left(x^{\beta}/n^{\beta\nu}\right)\right]^n$$

For $\nu = 1/\beta$ we then obtain

$$P\left(n^{1/\beta}(1 - X_{(n)}) \le x\right) = 1 - \left[1 - \left(x^{\beta}/n\right)\right]^n \to 1 - e^{-x^{\beta}}$$

It follows that $n^{1/\beta}(1 - X_{(n)}) \to T$ in distribution, where $T \sim \text{Weibull}(\beta, 1)$; cf. page 102 and exercise 3.26

Exercise 5.42.b

Let $X_1, X_2, ...$ be iid $\text{exponential}(1)$

We have $f_X(x) = e^{-x}$ (for $x > 0$) and

$$F_X(x) = \int_0^x e^{-u} du \quad = 1 - e^{-x}$$

Then we have

$$P\left(X_{(n)} - a_n \le x\right) = P\left(X_{(n)} \le x + a_n\right)$$

$$= P\left(X_i \le x + a_n \text{ for } i = 1, ..., n\right) = \left(1 - e^{-(x + a_n)}\right)^n$$

For $a_n = \log n$ we then obtain

$$P\left(X_{(n)} - \log n \le x\right) = \left(1 - e^{-x}/n\right)^n \to 1 - \exp(e^{-x})$$

It follows that $X_{(n)} - \log n \to Y$ in distribution, where Y has the extreme value distribution

Exercise 5.44

Let $X_1, X_2, \ldots.$ be iid Bernoulli random variables with success probability p

a) We have

$$\mathrm{E}X_i = p \quad \text{and} \quad \mathrm{Var}X_i = p(1-p)$$

By the central limit theorem (CLT) we then have

$$\frac{\sqrt{n}\left(Y_n - p\right)}{\sqrt{p(1-p)}} \to n(0,1) \text{ in distribution}$$

and hence (formally by Slutsky's theorem)

$$\sqrt{n}\left(Y_n - p\right) \to n(0, p(1-p)) \text{ in distribution}$$

b) We now consider

$$\sqrt{n}\left(Y_n(1-Y_n) - p(1-p)\right) = \sqrt{n}\left(g(Y_n) - g(p)\right)$$

where $g(x) = x(1-x) = x - x^2$

Note that

$$g'(x) = 1 - 2x$$

Now by the delta method (when $p \neq 1/2$)

$$\sqrt{n}\left(g(Y_n) - g(p)\right) \to n\left(0, \left[g'(p)\right]^2 p(1-p)\right)$$

Hence we have

$$\sqrt{n}\left(Y_n(1-Y_n) - p(1-p)\right) \to n\left(0, (1-2p)^2 p(1-p)\right)$$

c) Note that

$$g'(1/2) = 1 - 2(1/2) = 0$$

and that

$$g''(x) = -2$$

Then by the second order delta method we have

$$n\left(g(Y_n) - g(1/2)\right) \to \frac{1}{2}\left(1 - \frac{1}{2}\right)\frac{g''(1/2)}{2}\,\chi_1^2$$

Hence we have

$$n\left(Y_n(1-Y_n) - \frac{1}{4}\right) \to -\frac{1}{4}\,\chi_1^2$$