Solutions to exercises - Week 46

Most powerful tests

- Exercises 8.15, 8.22ac, 8.25bc, 8.31, 8.34b

Likelihood ratio tests

- Exercise 10.34a
- Additional exercise

Exercise 8.15

Let $X_1,....,X_n$ be iid and $n(0,\sigma^2)$

We will derive the most powerful test of

$H_0:\sigma=\sigma_0$ versus $H_1:\sigma=\sigma_1$ where $\sigma_0<\sigma_1$

The joint pdf of $\mathbf{X}=(X_1,....,X_n)$ is given by

$$f(\mathbf{x}\mid\sigma)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}$$

$$=\left(2\pi\sigma^2\right)^{-n/2}\exp\left\{-\sum_{i=1}^{n}x_i^2\Big/(2\sigma^2)\right\}$$

From Neyman-Pearsons Lemma, the most powerful test rejects H_0 if

$$\frac{f(\mathbf{x}\mid\sigma_1)}{f(\mathbf{x}\mid\sigma_0)}>k$$

Now we have

$$\frac{f(\mathbf{x}\mid\sigma_1)}{f(\mathbf{x}\mid\sigma_0)}=\frac{\left(2\pi\sigma_1^2\right)^{-n/2}\exp\left\{-\sum_{i=1}^{n}x_i^2\Big/(2\sigma_1^2)\right\}}{\left(2\pi\sigma_0^2\right)^{-n/2}\exp\left\{-\sum_{i=1}^{n}x_i^2\Big/(2\sigma_0^2)\right\}}$$

$$=\left(\frac{\sigma_0}{\sigma_1}\right)^n\exp\left\{\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n}x_i^2\right\}$$

Thus the most powerful test rejects H_0 if

$$\left(\frac{\sigma_0}{\sigma_1}\right)^n\exp\left\{\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n}X_i^2\right\}>k$$

i.e. if (since $\sigma_0<\sigma_1$)

$$\sum_{i=1}^{n}X_i^2>\frac{2\log\left[k(\sigma_1/\sigma_0)^n\right]}{1/\sigma_0^2-1/\sigma_1^2}=c$$

When H_0 is true, we have

$$\sum_{i=1}^{n}\left(X_i/\sigma_0\right)^2\sim\chi_n^2$$

Define $\chi^2_{n,\alpha}$ by $P\left(\chi^2_n > \chi^2_{n,\alpha}\right) = \alpha$

Then

$$\alpha = P\left(\sum_{i=1}^n \left(X_i / \sigma_0\right)^2 > \chi^2_{n,\alpha}\right) = P\left(\sum_{i=1}^n X_i^2 > \sigma_0^2\, \chi^2_{n,\alpha}\right)$$

So the test has size α if we let $c = \sigma_0^2\, \chi^2_{n,\alpha}$

Exercise 8.22

Let $X_1, ..., X_{10}$ be iid Bernoulli(p)

a) We will find most powerful test with size $\alpha = 0.0547$ for testing $H_0: p = 1/2$ versus $H_1: p = 1/4$

We have the pmf

$$f(\mathbf{x} \mid p) = \prod_{i=1}^{10} p^{x_i} (1-p)^{1-x_i} = p^y (1-p)^{10-y}$$

where $y = \sum x_i$

The most powerful test rejects H_0 if

$$\frac{f(\mathbf{x} \mid p = 1/4)}{f(\mathbf{x} \mid p = 1/2)} > k$$

Now we have

$$\frac{f(\mathbf{x} \mid p = 1/4)}{f(\mathbf{x} \mid p = 1/2)} = \frac{(1/4)^y (1-1/4)^{10-y}}{(1/2)^y (1-1/2)^{10-y}} = \left(\frac{3}{2}\right)^{10} \left(\frac{1}{3}\right)^y$$

Since the ratio is decreasing in y, we reject H_0 if $y \le c$

E.g. by using R [command: `pbinom(2,10,1/2)`] we find that $P(Y \le 2 \mid p = 1/2) = 0.0547$, so the most powerful test rejects H_0 if $Y \le 2$

The power of the test is

$$P(Y \le 2 \mid p = 1/4) = 0.526$$

c) There exist a most powerful test for all levels α that are given by

$$\alpha = P(Y \le c \mid p = 1/2) \quad \text{for} \quad c = 0, 1, ..., 10$$

Using R we find [command: `pbinom(0:10,10,1/2)`] that α can take the values

0.00098	0.0107	0.0547	0.1719	0.3770	0.6230
0.8281	0.9453	0.9893	0.9990	1.0000	

Exercise 8.25

b) Assume that T is Poisson(θ)

We have the pmf

$$g(t\,|\,\theta)=\frac{\theta^t}{t!}e^{-\theta}$$

For $\theta_2>\theta_1$ we have the likelihood ratio

$$\frac{g(t\,|\,\theta_2)}{g(t\,|\,\theta_1)}=\frac{\theta_2^t e^{-\theta_2}/t!}{\theta_1^t e^{-\theta_1}/t!}=\left(\frac{\theta_2}{\theta_1}\right)^t e^{\theta_1-\theta_2}$$

which is increasing in t

c) Assume that T is binomial(n,θ) with n known

We have the pmf

$$g(t\,|\,\theta)=\binom{n}{t}\theta^t(1-\theta)^{n-t}$$

For $\theta_2>\theta_1$ we have the likelihood ratio

$$\frac{g(t\,|\,\theta_2)}{g(t\,|\,\theta_1)}=\frac{\binom{n}{t}\theta_2^t(1-\theta_2)^{n-t}}{\binom{n}{t}\theta_1^t(1-\theta_1)^{n-t}}=\left(\frac{\theta_2(1-\theta_1)}{\theta_1(1-\theta_2)}\right)^t\left(\frac{1-\theta_2}{1-\theta_1}\right)^n$$

which is increasing in t

Exercise 8.31

Let $X_1, X_2, ..., X_n$ be iid $\text{Poisson}(\lambda)$

a) We will find a UMP test of $H_0:\lambda\le\lambda_0$ vs $H_1:\lambda>\lambda_0$

$T=\sum X_i\sim\text{Poisson}(n\lambda)$ is a sufficient statistic for λ and by exercise 8.25b it has an increasing likelihood ratio

By theorem 8.3.17 we then have that the UMP level α test rejects H_0 if $\sum X_i>k$ where $\alpha=P_{\lambda_0}\left(\sum X_i>k\right)$

b) We consider the case when $\lambda_0=1$

By the central limit theorem we have that

$$\frac{\sum X_i-n\lambda}{\sqrt{n\lambda}}\to Z\sim n(0,1)$$

Therefore

$$P\left(\sum X_i>k\,|\,\lambda=1\right)=P\left(\frac{\sum X_i-n}{\sqrt{n}}>\frac{k-n}{\sqrt{n}}\,|\,\lambda=1\right)\approx P\left(Z>\frac{k-n}{\sqrt{n}}\right)$$

and

$$P\left(\sum X_i>k\,|\,\lambda=2\right)=P\left(\frac{\sum X_i-2n}{\sqrt{2n}}>\frac{k-2n}{\sqrt{2n}}\,|\,\lambda=2\right)$$

$$\approx P\left(Z>\frac{k-2n}{\sqrt{2n}}\right)$$

We obtain an approximate 5% level test with power approximately 90% for $\lambda = 2$ if we chose n such that

$$P\left(Z > \frac{k-n}{\sqrt{n}}\right) = 0.05 \quad \text{and} \quad P\left(Z > \frac{k-2n}{\sqrt{2n}}\right) = 0.90$$

This gives

$$\frac{k-n}{\sqrt{n}} = 1.645 \qquad \frac{k-2n}{\sqrt{2n}} = -1.28$$

and we obtain

$$n = 12 \quad \text{and} \quad k = 17.7$$

Exercise 8.34.b

We assume that the family $\{g(t\,|\,\theta) : \theta \in \Theta\}$ of pdfs or pmfs for T has a nondecreasing likelihood ratio, i.e. that for every $\theta_2 > \theta_1$ the ratio $g(t\,|\,\theta_2)/g(t\,|\,\theta_1)$ is a nondecreasing function of t on the set $\{t: g(t\,|\,\theta_1) > 0 \text{ or } g(t\,|\,\theta_2) > 0\}$

We will show that for $\theta_2 > \theta_1$ we have

$$P_{\theta_1}(T > c) \le P_{\theta_2}(T > c)$$

or equivalently

$$P_{\theta_2}(T \le c) \le P_{\theta_1}(T \le c)$$

We will show the result for pdfs (the argument for pmfs is similar, but with sums instead of integrals)

We consider the function

$$D(c) \overset{\text{def}}{=} P_{\theta_2}(T \le c) - P_{\theta_1}(T \le c) = \int_{-\infty}^{c} g(t\,|\,\theta_2)dt - \int_{-\infty}^{c} g(t\,|\,\theta_1)dt$$

$$= \int_{-\infty}^{c} \left[g(t\,|\,\theta_2)/g(t\,|\,\theta_1) - 1\right] g(t\,|\,\theta_1)dt$$

Then $D'(c) = \left[g(c\,|\,\theta_2)/g(c\,|\,\theta_1) - 1\right] g(c\,|\,\theta_1)$

Now $g(c\,|\,\theta_2)/g(c\,|\,\theta_1)$ is increasing, so $D'(c)$ can only change sign from negative to positive showing that any interior extremum of $D(c)$ is a minimum.

Thus $D(c)$ is maximized by its value at $-\infty$ or ∞, which is zero.

Hence $D(c) \le 0$, and the result is proved

Exercise 10.34.a

Let $X_1, ..., X_n$ be iid Bernoulli(p)

We will test $H_0: p = p_0$ versus $H_1: p \ne p_0$

The likelihood is given by [$\mathbf{x} = (x_1, ..., x_n)$]

$$L(p\,|\,\mathbf{x}) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^y(1-p)^{n-y}$$

where $y = \sum_{i=1}^{n} x_i$

The unrestricted MLE for p is $\hat{p} = y/n$

The LRT statistic becomes

$$\lambda(\mathbf{x}) = \frac{L(p_0 \mid \mathbf{x})}{L(\hat{p} \mid \mathbf{x})} = \frac{p_0^y (1-p_0)^{n-y}}{\hat{p}^y (1-\hat{p})^{n-y}}$$

Hence we have

$$-2\log\lambda(\mathbf{x}) = 2\log\left(\frac{\hat{p}^y (1-\hat{p})^{n-y}}{p_0^y (1-p_0)^{n-y}}\right)$$

$$= 2y\log\left(\frac{\hat{p}}{p_0}\right) + 2(n-y)\log\left(\frac{1-\hat{p}}{1-p_0}\right)$$

Additional exercise

Let $\mathbf{X}_i = (X_{i1}, X_{i2}, X_{i3});\ i = 1, 2, ..., n$ be iid with pmf

$$f(x_1, x_2, x_3 \mid p_1, p_2, p_3) = p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_i \in \{0,1\}$ with $x_1 + x_2 + x_3 = 1$ and $p_i \geq 0$ with $p_1 + p_2 + p_3 = 1$

We will test (Hardy-Weinberger)

$$H_0: p_1 = \theta^2,\ p_2 = 2\theta(1-\theta),\ p_3 = (1-\theta)^2$$

for a $\theta \in (0,1)$ versus the alternative that H_0 does not hold

The likelihood is given by $[\ \mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)\ ]$

$$L(p_1, p_2, p_3 \mid \mathbf{x}) = \prod_{i=1}^{n} p_1^{x_{i1}} p_2^{x_{i2}} p_3^{x_{i3}} = p_1^{y_1} p_2^{y_2} p_3^{y_3}$$

where $y_j = \sum_{i=1}^{n} x_{ij}$ for $j = 1, 2, 3$

Using the facts that $p_3 = 1 - p_1 - p_2$ and $y_3 = n - y_1 - y_2$ one may show that the MLE of the p_j's are $\hat{p}_j = y_j / n$

Under H_0 we may write the likelihood as

$$L(\theta \mid \mathbf{x}) = \left[\theta^2\right]^{y_1} \left[2\theta(1-\theta)\right]^{y_2} \left[(1-\theta)^2\right]^{y_3}$$

$$= 2^{y_2} \theta^{2y_1 + y_2} (1-\theta)^{2y_3 + y_2}$$

$$= 2^{y_2} \theta^{2y_1 + y_2} (1-\theta)^{2n - (2y_1 + y_2)}$$

One may then show that the MLE of θ is

$$\theta^* = \frac{2y_1 + y_2}{2n}$$

and hence that the MLEs of the p_j's under H_0 are

$$p_1^* = (\theta^*)^2, \quad p_2^* = 2\theta^*(1-\theta^*) \quad \text{and} \quad p_2^* = (1-\theta^*)^2$$

Then we find

$$-2\log\lambda(\mathbf{x}) = -2\log\left(\frac{\left(p_1^*\right)^{y_1} \left(p_2^*\right)^{y_2} \left(p_3^*\right)^{y_3}}{\hat{p}_1^{y_1} \hat{p}_2^{y_2} \hat{p}_3^{y_3}}\right)$$

$$= 2\sum_{j=1}^{3} y_j \log\left(\frac{\hat{p}_j}{p_j^*}\right)$$