STK4011 and STK9011
Autumn 2016

Multivariate distributions

Covers (most of) the following material from chapter 4:

- Section 4.6: pages 177-178 and 182-186

Ørnulf Borgan
Department of Mathematics
University of Oslo

We consider a continuous multivariate random vector $\mathbf{X} = (X_1, \ldots, X_n)$

For any set $A \subset \mathcal{R}^n$ we have that

$$P(\mathbf{X} \in A) = \int \ldots \int_A f(\mathbf{x}) d\mathbf{x} = \int \ldots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is the joint pdf

The marginal pdf of $(X_1, \ldots, X_k)$ is given by

$$f(x_1, \ldots, x_k) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_{k+1} \ldots dx_n$$

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a multivariate random vector with joint pdf $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ and let $f_{X_i}(x_i)$ denote the marginal pdf of X_i; $i = 1, \ldots, n$

Then $X_1, \ldots, X_n$ are mutually independent random variables if for $(x_1, \ldots, x_n) \in \mathcal{R}^n$ we may write

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

Theorem 4.6.11
$X_1, \ldots, X_n$ are mutually independent if and only if there exist functions $g_i(x_i)$, $i = 1, \ldots, n,$ such that for all $(x_1, \ldots, x_n) \in \mathcal{R}^n$ we may write

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} g_i(x_i)$$

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a multivariate random vector with joint pdf $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ and let $g(\mathbf{x}) = g(x_1, \ldots, x_n)$ be a real valued function

Then the expected value of $g(\mathbf{X})$ is given by

$$\begin{aligned} \mathrm{E}g(\mathbf{X}) &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(\mathbf{x}) f(\mathbf{x})\, d\mathbf{x} \\ &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(x_1, \ldots, x) f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \end{aligned}$$

If $X_1, \ldots, X_n$ are mutually independent random variables and $g_i(x_i)$, $i = 1, \ldots, n,$ are real valued functions such that $g_i(x_i)$ is a function only of x_i, then

$$\mathrm{E}\left(\prod_{i=1}^{n} g_i(X_i) \right) = \prod_{i=1}^{n} \mathrm{E}\, g_i(X_i)$$

Let $X_1,\ldots,X_n$ be mutually independent random variables with mgfs $M_{X_i}(t)$, $i=1,\ldots,n,$ and let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be constants. Then the mgf of $Z=\sum_{i=1}^{n}(a_iX_i+b_i)$ is

$$M_Z(t)=\exp\left(t\sum b_i\right)\prod_{i=1}^{n}M_{X_i}(a_i t)$$

This result may be used to prove that if $X_1,\ldots,X_n$ are independent and $X_i\sim n(\mu_i,\sigma_i^2)$, then

$$Z=\sum_{i=1}^{n}(a_1X_1+b_i)\sim n\left(\sum a_i\mu_i+b_i\ ,\ \sum a_i^2\sigma_i^2\right)$$

Multivariate transformations

Let $\mathbf{X}=(X_1,\ldots,X_n)$ be a multivariate random vector with joint pdf $f_{\mathbf{X}}(\mathbf{x})=f_{\mathbf{X}}(x_1,\ldots,x_n)$ and support

$$\mathcal{A}=\{\mathbf{x}:\ f_{\mathbf{X}}(\mathbf{x})>0\}$$

Consider a new random vector $\mathbf{U}=(U_1,\ldots,U_n)$ defined by $U_i=g_i(\mathbf{X})=g_i(X_1,\ldots,X_n)$, $i=1,\ldots,n$

The transformation $(u_1,\ldots,u_n)=(g_1(\mathbf{x}),\ldots,g_n(\mathbf{x}))$ is a one-to-one transformation from $\mathcal{A}$ onto

$$\mathcal{B}=\{\mathbf{u}:\mathbf{u}=(u_1,\ldots,u_n)=(g_1(\mathbf{x}),\ldots,g_1(\mathbf{x}))\text{ for }\mathbf{x}\in\mathcal{A}\}$$

The inverse transformation is given by

$x_i=h_i(\mathbf{u})=h_i(u_1,\ldots,u_n)$, $i=1,\ldots,n$

Then the joint pdf of $\mathbf{U}=(U_1,\ldots,U_n)$ is given by (for $\mathbf{u}\in\mathcal{B}$):

$$f_{\mathbf{U}}(u_1,\ldots,u_n)=f_{\mathbf{X}}(h_1(u_1,\ldots,u_n),\ldots,h_n(u_1,\ldots,u_n))\left|J(u_1,\ldots,u_n)\right|$$

Here $J(u_1,\ldots,u_n)$ is the Jacobian:

$$J(u_1,\ldots,u_n)=\begin{vmatrix}\frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n}\\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n}\end{vmatrix}$$

Example 4.6.13

We have the joint pdf

$$f_{\mathbf{X}}(x_1,x_2,x_3,x_4)=\begin{cases}24e^{-x_1-x_2-x_3-x_4} & 0<x_1<x_2<x_3<x_4<\infty\\ 0 & \text{otherwise}\end{cases}$$

Consider the transformation

$$U_1=X_1 \qquad U_2=X_2-X_1$$
$$U_3=X_3-X_2 \qquad U_4=X_4-X_3$$

The support of the distribution of the U_i s is

$$\mathcal{B}=\{(u_1,u_2,u_3,u_4):u_i>0,\ i=1,2,3,4\}$$

The inverse transformation is

$$X_1 = U_1 \qquad X_2 = U_1 + U_2$$
$$X_3 = U_1 + U_2 + U_3 \qquad X_4 = U_1 + U_2 + U_3 + U_4$$

The Jacobian becomes

$$J(u_1,u_2,u_3,u_4) = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} & \frac{\partial x_1}{\partial u_4} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} & \frac{\partial x_2}{\partial u_4} \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3} & \frac{\partial x_3}{\partial u_4} \\ \frac{\partial x_4}{\partial u_1} & \frac{\partial x_4}{\partial u_2} & \frac{\partial x_4}{\partial u_3} & \frac{\partial x_4}{\partial u_4} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1$$

The joint pdf of the U_i s becomes (when all $u_i > 0$)

$$f_{\mathbf{U}}(u_1,u_2,u_3,u_4) = 24e^{-u_1 - (u_1+u_2) - (u_1+u_2+u_3) - (u_1+u_2+u_3+u_4)}$$
$$= 24e^{-4u_1 - 3u_2 - 2u_3 - u_4}$$
$$= 4e^{-4u_1}\, 3e^{-3u_2}\, 2e^{-2u_3}\, e^{-u_4}$$

The U_i s are independent and

$$U_i \sim \text{exponential}(1/(5-i)), \quad i = 1,2,3,4$$