Supplementary exercises in STK4011/STK9011

Problem 1

Consider the exponential family where the pdf/pmf have the standard form

$$f(x|\eta) = h(x)c^*(\eta)\exp(\Sigma_{i=1}^k \eta_i t_i(x))$$

a) Show that the natural parameter space is convex.

b) Show that $1/c(\eta)$ is a convex function.

Problem 2

Consider the exponential family where the pdf/pmf have the standard form

$$f(x|\eta) = h(x)c^*(\eta)\exp(\Sigma_{i=1}^k \eta_i t_i(x))$$

Prove that, under the assumption that differentiation and integration can be interchanged,

a) $E[t_i(X)] = -\partial/\partial\eta_i \log(c^*(\eta)),\ i = 1, \cdots, k$

b) $Cov(t_i(X), t_j(X) = -\frac{\partial^2}{\partial\eta_i \partial/\eta_j} \log(c^*(\eta)),\ i, j = 1, \cdots, k$

Problem 3

Assume that the random variable X has pdf

$$f(x|\theta) = (x/\theta^2)\exp(-x^2/2\theta^2),\ x > 0, \theta > 0$$

which defines the *Rayleigh* distribution. Show that the distribution belongs to the exponential family of distributions and compute the expectation and variance of X^2.

Problem 4

Consider the exponential family where the pdf/pmf have the standard form

$$f(x|\eta) = h(x)c^*(\theta)\exp(\Sigma_{i=1}^k \theta_i t_i(x))$$

Find an expression for the the moment generating function $E[\exp(\Sigma_{i=1}^{k} u_i t_i(x))]$, and verify that you get the same expectation and variance as in problem 2

Problem 5

Consider the gamma distribution with probability density function

$$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp(-x/\beta),\ 0 < x < \infty,\ \alpha > 0, \beta > 0.$$

a) Write the pdf of gamma distribution on the standard form and find the natural parameters η_1 and η_2 and the statistics T_1 and T_2.

b) Express the expectations of $\log(X)$ by the natural parameter and vice versa.

Problem 6

Let X and Y be independent, $X \sim U[0,1]$ and $Y \sim U[0,1]$. Find the pdf, probability density function, and mgf, moment generating function, of $X+Y$.

Problem 7

Consider a discrete sample $\mathbf{X}$ with probability mass function $f(\mathbf{x}|\theta)$. Show that if $T(\mathbf{X})$ is a sufficient and complete statistic and $U(\mathbf{X})$ is a minimal sufficient statistic, then $P(U(\mathbf{X}) = U(\mathbf{x})) = P(T(\mathbf{X}) = T(\mathbf{x}))$ for all $\mathbf{x}$ such that $P(\mathbf{X} = \mathbf{x}) > 0$.