Assignment in STK4011/STK9011-f17

Posted: Friday November 24th at 9 am.

Deadline: Tuesday November 28th at 2pm.

This is text for the final assignments in STK4011/STK9011-f17. It consists of three problems. Both handwritten reports and answers using a word processing system are acceptable. Each student must deliver a separate and individually formulated report. That means that you should not compare your answers, and discussion of the problems should only involve general viewpoints on how the problems can be solved. There is more information at: http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html

There are two ways to deliver the reports. You can choose what is most convenient.

1. You can deliver **two** paper copies with your candidate number at the Math department's expedition at Ullevål. Remember to indicate whether you attend STK4011 or STK9011.

2. Use the electronic filing system, delivery, for uploading exam papers.

Problem 1

Let X, Y and Z be independent and uniformly distributed on the interval $[0, 1]$.

a) Use the convolution formula to find the density of $X + Y$.

b) Verify the results from part a) by using moment generating functions.

c) Use the convolution formula to find the density of $X + Y + Z$.

Problem 2

The Pareto distribution is often used to model phenomena with heavy tails such as income. The density is

$$f(x|\alpha, \beta) = \begin{cases} \frac{\beta\alpha^\beta}{x^{\beta+1}} & \text{if } \alpha < x < \infty \quad \alpha, \beta > 0 \\ 0 & \text{else} \end{cases} \tag{1}$$

Assume that $X_1, \ldots, X_n$ is a random sample of random variables which are Pareto distributed so the probability density is $f(x|\alpha, \beta)$.

a) Show that the cumulative distribution function equals

$$F_X(x) = 1 - (\frac{\alpha}{x})^\beta \text{ for } x > \alpha.$$

b) Find $E[X^k]$ when X is Pareto distributed, i.e has density (1).

c) Show that the Pareto distribution belongs to the exponential family of distributions when α is a known constant. Also show that $\log X - \log \alpha$ has an exponential distribution with scale parameter $1/\beta$.

d) Show that the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ of β and α are

$$\hat{\beta} = \frac{n}{\Sigma_{i=1}^{n} \log(X_i/X_{(1)})} \text{ and } \hat{\alpha} = X_{(1)}$$

where $X_{(1)} = \min_i X_i$.

e) Show that $\hat{\beta}$ and $\hat{\alpha}$ are independently distributed and that $2n\beta/\hat{\beta}$ is χ^2 distributed with $2(n-1)$ degrees of freedom and $\hat{\alpha}$ is Pareto distributed.

[Hint: To show independence argue that it is sufficient to consider independent and exponentially distributed random variables $U_1, \dots, U_n$. Then use the transformation $V_1 = U_{(1)}$ and $V_i = (n-i+1)(U_{(i)} - U_{(1)}), i = 2, \dots, n$ where $(U_{(1)}, \dots, U_{(n)})$ is the order statistic. Before treating the general case it may be helpful to consider the situation where n=3 or 4.]

f) Show that $X_{(1)}$ and $\Sigma_{i=1}^{n} \log(X_i/X_{(1)})$ are sufficient statistics for α and β based on a random sample where the probability density is Pareto , i.e. has a probability distribution defined in equation (1).

It can also be shown the statistics $X_{(1)}$ and $\Sigma_{i=1}^{n} \log(X_i/X_{(1)})$ are complete statistics for α and β based on a random sample where the probability density is Pareto. Assuming that:

g) Find the UMVUE or best unbiased estimator for β.

h) Also show that $X_{(1)}[1 - \frac{1}{(n-1)}\frac{1}{\hat{\beta}}]$ is the UMVUE or best unbiased estimator for α.

Problem 3

Let $X_1, \dots, X_n$ be a random sample from the logistic distribution which has cumulative distribution function

$$F_X(x) = \frac{1}{1 + \exp(-(x-\theta))}, \; -\infty < x < \infty, \; -\infty < \theta < \infty.$$

Here θ is an unknown parameter.

a) Show that the empirical mean

$$\theta_n^* = \frac{1}{n}\Sigma_{i=1}^n X_i$$

is the method of moment estimator for θ. What is the approximate distribution of θ_n^* for large n? Here you can use without proof that $Var(X_1) = \pi^2/3$.

b) Find an equation satisfied by the maximum likelihood estimator $\hat{\theta}_n$. Show that $\hat{\theta}_n$ exists and is unique.

c) What is the approximate distribution of $\hat{\theta}_n$ for large n? What is the asymptotic relative efficiency, ARE, of θ_n^* with respect to $\hat{\theta}_n$?

d) Based on one observation, X_1 say, describe the uniformly most powerful test with level α for the null hypothesis

$$H_0 : \theta = \theta_0 \text{ versus the alternative hypothesis } H_1 : \theta > \theta_0.$$

e) Explain why the test from part d) also is the uniformly most powerful test with level α for the null hypothesis

$$H_0 : \theta \leq \theta_0 \text{ versus the alternative hypothesis } H_1 : \theta > \theta_0.$$