# Compulsory assignment in STK4011/STK9011-f18

Posted: Wednesday October 3rd at 10 am.

Deadline: Thursday October 25th at 2.30 pm.

This is text for the assignments in STK4011/STK9011-f18. It consists of three problems. Both handwritten reports and answers using a word processing system are acceptable.

Each student must deliver a separate and individually formulated report, but discussions with fellow students how to solve the problems are OK. There is more information at:
http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html

There are two ways to deliver the reports. You can choose what is most convenient.

1. You can deliver a paper copy with your name/candidate number at the Math department's expedition at 7th floor in N.H. Abel's house. Remember to indicate whether you attend STK4011 or STK9011.

2. Use the electronic filing system, devilry, for uploading exam papers.

### Problem 1

Let $X, Y$ and $Z$ be independent and uniformly distributed on the interval $[0, 1]$.

a) Use the convolution formula to find the density of $X + Y$.

b) Verify the results from part a) by using moment generating functions.

c) Use the convolution formula to find the density of $X + Y + Z$.

### Problem 2

The Pareto distribution is often used to model phenomena with heavy tails such as income. The density is

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta\alpha^\beta}{x^{\beta+1}} & \text{if } \alpha \le x < \infty \quad \alpha, \beta > 0 \\ 0 & \text{else} \end{cases} \tag{1}$$

Assume that $X_1, \ldots, X_n$ is a random sample of random variables which are Pareto distributed so the probability density is $f(x|\alpha, \beta)$.

a) Show that the cumulative distribution function equals

$$F_X(x) = 1 - (\frac{\alpha}{x})^\beta \text{ for } x \ge \alpha.$$

b) Find $E[X^k]$ when X is Pareto distributed, i.e has density (1).

c) Show that the Pareto distribution belongs to the exponential family of distributions when $\alpha$ is a known constant. Also show that $\log X - \log \alpha$ has an exponential distribution with scale parameter $1/\beta$.

d) Show that the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ of $\beta$ and $\alpha$ are

$$\hat{\beta} = \frac{n}{\Sigma_{i=1}^{n} \log(X_i/X_{(1)})} \text{ and } \hat{\alpha} = X_{(1)}$$

where $X_{(1)} = \min_i X_i$.

e) Show that $\hat{\beta}$ and $\hat{\alpha}$ are independently distributed and that $2n\beta/\hat{\beta}$ is $\chi^2$ distributed with $2(n-1)$ degrees of freedom and $\hat{\alpha}$ is Pareto distributed.

[Hint: To show independence argue that it is sufficient to consider independent and exponentially distributed random variables $U_1, \ldots, U_n$. Then use the transformation $V_1 = U_{(1)}$ and $V_i = (n-i+1)(U_{(i)} - U_{(1)}), i = 2, \ldots, n$ where $(U_{(1)}, \ldots, U_{(n)})$ is the order statistic. Before treating the general case it may be helpful to consider the situation where n=3 or 4.]

f) Show that $X_{(1)}$ and $\Sigma_{i=1}^{n} \log(X_i/X_{(1)})$ are sufficient statistics for $\alpha$ and $\beta$ based on a random sample where the probability density is Pareto , i.e. has a probability distribution defined in equation (1).

It can also be shown the statistics $X_{(1)}$ and $\Sigma_{i=1}^{n} \log(X_i/X_{(1)})$ are complete statistics for $\alpha$ and $\beta$ based on a random sample where the probability density is Pareto. Assuming that:

g) Find the UMVUE or best unbiased estimator for $\beta$.

h) Also show that $X_{(1)}[1 - \frac{1}{(n-1)}\frac{1}{\hat{\beta}}]$ is the UMVUE or best unbiased estimator for $\alpha$.

**Problem 3**

Let $X_1, \ldots, X_n$ be a random sample from the geometric distribution which has probability mass function

$$f_X(x|p) = p(1-p)^{x-1},\ x = 1, 2, \ldots$$

Here $p$ is an unknown parameter.

a) Find the expectation and variance of the estimator $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

b) What is the Cramer Rao lower bound for the estimator $\bar{X}$. What can you conclude?

c) Find the maximum likelihood estimator (MLE), $\hat{p}$, for $p$.

d) Let $\tilde{p} = 1$ if $X_1 = 1$ and equal to 0 else. Explain why $\tilde{p}$ is an unbiased estimator for $p$. Use the Rao Blackwell technique to find a best unbiased estimator, BUE, for $p$.

e) Explain why the bias of the MLE $\hat{p}$ is negative.