Solutions to exercises - Week 36

Beta function:

- Additional exercise

Exponential family of distributions:

- Exercises 3.28b-d, 3.30b and 3.33b-c

Location/scale distributions:

- Exercises 3.38, and 3.39

Bivariate distributions:

- Exercises 4.4a-b and 4.5

Additional exercise

We have the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

and the beta function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx$$

We will prove the relation

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

We have to prove that

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta)$$

Now we may write

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx \int_0^\infty y^{\alpha-1}e^{-y}\,dy$$

$$= \int_0^\infty \int_0^\infty x^{\alpha-1}e^{-x}\, y^{\beta-1}e^{-y}\,dydx$$

$$= \int_0^\infty \int_0^\infty x^{\alpha-1}y^{\beta-1}e^{-(x+y)}\,dydx$$

We then perform a change of variables:

$$u = x+y \quad \text{and} \quad v = \frac{x}{x+y}$$

Note that $u > 0$ and $0 < v < 1$

This gives $x = uv$ and $y = u(1-v)$

The Jacobian becomes

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -vu - (1-v)u = -u$$

We then obtain

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^1 (uv)^{\alpha-1}[u(1-v)]^{\beta-1} e^{-u} \,|-u|\, dv du$$

$$= \int_0^\infty \int_0^1 u^{\alpha+\beta-1} e^{-u}\, v^{\alpha-1}(1-v)^{\beta-1}\, dv du$$

$$= \int_0^\infty u^{\alpha+\beta-1} e^{-u} du \int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\, dv \; = \Gamma(\alpha+\beta)B(\alpha,\beta)$$

Exercise 3.28b

$$f(x\mid\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^{\alpha}\,\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & \text{if } x>0\\ 0 & \text{otherwise}\end{cases}$$

For all x we may write:

$$f(x\mid\alpha,\beta)=\underbrace{I_{\{x>0\}}(x)}_{h(x)}\;\underbrace{\frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}}_{c(\alpha,\beta)}\;\exp\left\{\underbrace{(\alpha-1)}_{w_1(\alpha,\beta)}\;\underbrace{\log x}_{t_1(x)}+\underbrace{\left(-\frac{1}{\beta}\right)}_{w_2(\alpha,\beta)}\underbrace{x}_{t_2(x)}\right\}$$

Exercise 3.28c

$$f(x|\alpha,\beta)=\begin{cases}\dfrac{1}{B(\alpha,\beta)}\,x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0<x<1\\[2ex] 0 & \text{otherwise}\end{cases}$$

For all x we may write:

$$f(x|\alpha,\beta)=\underbrace{I_{(0,1)}(x)}_{h(x)}\,\underbrace{\frac{1}{B(\alpha,\beta)}}_{c(\alpha,\beta)}\exp\left(\underbrace{(\alpha-1)}_{w_1(\alpha,\beta)}\underbrace{\log x}_{t_1(x)}+\underbrace{(\beta-1)}_{w_2(\alpha,\beta)}\underbrace{\log(1-x)}_{t_2(x)}\right)$$

Exercise 3.28d

$$f(x\,|\,\lambda) = \begin{cases} \dfrac{\lambda^x}{x!} e^{-\lambda} & \text{for } x = 0, 1, 2, \ldots. \\ 0 & \text{otherwise} \end{cases}$$

For all x we may write:

$$f(x\,|\,\lambda) = \underbrace{I_{\{0,1,2,\ldots.\}}(x)\,\frac{1}{x!}}_{h(x)}\;\underbrace{e^{-\lambda}}_{c(\lambda)}\,\exp\Big(\underbrace{\log(\lambda)}_{w_1(\lambda)}\;\underbrace{x}_{t_1(x)}\Big)$$

Exercise 3.30b

From exercise 3.28.d we have:

$$f(x|\lambda) = \underbrace{I_{\{0,1,2,....\}}(x)\,\frac{1}{x!}}_{h(x)}\;\underbrace{e^{-\lambda}}_{c(\lambda)}\exp\left(\underbrace{\log(\lambda)}_{w_1(\lambda)}\;\underbrace{x}_{t_1(x)}\right)$$

The relations in Theorem 3.4.2 become

$$\mathrm{E}\left(\frac{1}{\lambda}X\right) = 1 \qquad \mathrm{Var}\left(\frac{1}{\lambda}X\right) = 0 - \mathrm{E}\left(-\frac{1}{\lambda^2}X\right)$$

From these relations we obtain

$$\mathrm{E}\,X = \lambda$$

$$\mathrm{Var}\;\;X\;\; = \lambda^2\left\{0 + \frac{1}{\lambda^2}\lambda\right\} = \lambda$$

Exercise 3.33.b

Assume $X \sim n(\theta, a\theta^2)$

The pdf takes the form

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}\sqrt{a\theta^2}} \exp\left(-\frac{(x-\theta)^2}{2a\theta^2}\right) \qquad a > 0 \quad \text{known}$$

This may be written

$$f(x|\theta) = \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{a\theta^2}} \exp\left(-\frac{1}{2a}\right)}_{c(\theta)} \exp\left(\underbrace{\frac{1}{2a\theta^2}}_{w_1(\theta)} \underbrace{-x^2}_{t_1(x)} + \underbrace{\frac{1}{a\theta}}_{w_2(\theta)} \underbrace{x}_{t_2(x)}\right)$$

For the normal distribution the full parameter space is

$$(\mu, \sigma^2): -\infty < \mu < \infty,\ \sigma^2 > 0$$

Here we have a curved exponential distribution with parameter space

$$(\mu, \sigma^2): \mu = \theta,\ \sigma^2 = a\theta^2,\ -\infty < \theta < \infty$$

Exercise 3.33.c

Assume $X \sim \text{gamma}(\alpha, 1/\alpha)$

The pdf takes the form

$$f(x|\alpha) = \begin{cases} \dfrac{\alpha^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

For all x we may write:

$$f(x|\alpha) = \underbrace{I_{\{x>0\}}(x)}_{h(x)} \underbrace{\frac{\alpha^{\alpha}}{\Gamma(\alpha)}}_{c(\alpha)} \exp\left(\underbrace{(\alpha - 1)}_{w_1(\alpha)} \underbrace{\log x}_{t_1(x)} + \underbrace{\alpha}_{w_2(\alpha)} \underbrace{(-x)}_{t_2(x)} \right)$$

For the gamma distribution the full parameter space is

$$(\alpha, \beta): \ \alpha > 0, \ \ \beta > 0$$

Here we have a curved exponential distribution with parameter space

$$(\alpha, \beta): \ \alpha > 0, \ \beta = 1/\alpha$$

Exercise 3.38

$$Z \sim f(z) \qquad \alpha = P(Z > z_\alpha) = \int_{z_\alpha}^{\infty} f(z)\,dz$$

$$X \text{ has pdf} \quad f_X(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

Then

$$\alpha = P(X > x_\alpha) = \int_{x_\alpha}^{\infty} f_X(x)\,dx = \int_{x_\alpha}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx = \int_{\frac{x_\alpha - \mu}{\sigma}}^{\infty} f \quad z \quad dz$$

Therefore

$$\frac{x_\alpha - \mu}{\sigma} = z_\alpha \quad \text{and} \quad x_\alpha = \sigma z_\alpha + \mu$$

Exercise 3.39

$$f_X(x) = \frac{1}{\sigma\pi} \frac{1}{1 + \left[(x-\mu)/\sigma\right]^2} \qquad \text{for} \quad -\infty < x < \infty$$

The pdf of $Z = (X-\mu)/\sigma$ is given by

$$f(z) = \frac{1}{\pi} \frac{1}{1+z^2} \qquad \text{for} \quad -\infty < z < \infty$$

Recall that $\dfrac{d}{du}\arctan(u) = \dfrac{1}{1+u^2}$

The cdf of Z becomes

$$F(z) = \int_{-\infty}^{z} \frac{1}{\pi} \frac{1}{1+u^2}\, du = \frac{1}{\pi} \arctan(u) \Big|_{-\infty}^{z} = \frac{1}{\pi}\arctan(z) + \frac{1}{2}$$

We have

$$P(Z \le 0) = F(0) = \frac{1}{\pi}\arctan(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

Further we have

$$P(Z \le -1) = F(-1) = \frac{1}{\pi}\arctan(-1) + \frac{1}{2} = \frac{1}{\pi}\left(-\frac{\pi}{4}\right) + \frac{1}{2} = \frac{1}{4}$$

$$P(Z \ge 1) = 1 - F(1) = \frac{1}{2} - \frac{1}{\pi}\arctan(1) = \frac{1}{2} - \frac{1}{\pi}\frac{\pi}{4} = \frac{1}{4}$$

Finally for $X = \sigma Z + \mu$ we have

$$P(X \le x) = P(Z \le (x-\mu)/\sigma) = F((x-\mu)/\sigma)$$

$$P(X \le \mu) = F(0) = 1/2$$

$$P(X \le \mu - \sigma) = F(-1) = 1/4$$

$$P(X \ge \mu + \sigma) = 1 - P(X \le \mu + \sigma) = 1 - F(1) = 1/4$$

Exercise 4.4.a

$$f(x,y)=\begin{cases} C(x+2y) & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

We have:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^2 C(x+2y)\,dx\,dy = C\int_0^1\left[\frac{x^2}{2}+2yx\right]_0^2 dy$$

$$= C\int_0^1 2+4y \ \ dy = C\left[2y+2y^2\right]_0^1 = 4C$$

From this we obtain $C=\dfrac{1}{4}$

Exercise 4.4.b

$$f(x,y)=\begin{cases}\frac{1}{4}(x+2y) & 0<x<2,\ 0<y<1\\ 0 & \text{otherwise}\end{cases}$$

Marginal distribution of X (for $0 < x < 2$)

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{1}{4}(x+2y)\,dy$$

$$=\left[\frac{1}{4}(xy+y^2)\right]_0^1 = \frac{1}{4}(x+1)$$

Exercise 4.5.a

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$P(X > \sqrt{Y}) = P(Y < X^2) = \int_0^1 \int_0^{x^2} (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{x^2} dx = \int_0^1 \left(x^3 + \frac{x^4}{2} \right) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 = \frac{7}{20}$$

Exercise 4.5.b

$$f(x,y)=\begin{cases} 2x & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$P(X^2<Y<X) = \int_0^1 \int_{x^2}^{x} 2x\,dy\,dx = \int_0^1 2xy\Big|_{x^2}^{x} dx$$

$$= \int_0^1 2x^2 - 2x^3\ dx = \left[\frac{2x^3}{3} - \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{6}$$