Solutions to exercises - Week 39

Sufficient statistic:

- Exercises 6.2, 6.3, 6.4, 6.5, 6.6 and 6.7

Minimal sufficient statistics:

- Exercises 6.8 and 6.9a-c

Exercise 6.2

$X_1,\dots,X_n$ are independent with pdfs

$$f_{X_i}(x_i\mid\theta)=\begin{cases} e^{i\theta-x_i} & \text{for } x_i\ge i\theta \\ 0 & \text{otherwise}\end{cases}$$

Note that for all x_i we may write

$$f_{X_i}(x_i\mid\theta)=e^{i\theta-x_i}I_{[i\theta,\infty)}(x_i)=e^{i\theta-x_i}I_{[\theta,\infty)}(x_i/i)$$

Hence the joint pdf may be written

$$f(\mathbf{x}\mid\theta)=\prod_{i=1}^n e^{i\theta-x_i}I_{[\theta,\infty)}(x_i/i)=\underbrace{e^{\theta\sum i}I_{[\theta,\infty)}(\min_i(x_i/i))}_{g(\min_i(x_i/i)\mid\theta)}\;\underbrace{e^{-\sum x_i}}_{h(\mathbf{x})}$$

Thus $T(\mathbf{X})=\min_i(X_i/i)$ is sufficient for θ

Exercise 6.3

$X_1,\dots,X_n$ are iid with pdf

$$f(x\mid\mu,\sigma)=\begin{cases} (1/\sigma)e^{-(x-\mu)/\sigma} & \mu<x<\infty \\ 0 & \text{otherwise}\end{cases}$$

Note that we may write

$$f(x\mid\mu,\sigma)=(1/\sigma)e^{-(x-\mu)/\sigma}I_{(\mu,\infty)}(x)$$

Hence the joint pdf may be written

$$f(\mathbf{x}\mid\mu,\sigma)=\prod_{i=1}^n(1/\sigma)e^{-(x_i-\mu)/\sigma}I_{(\mu,\infty)}(x_i)\;=\frac{e^{n\mu/\sigma}}{\sigma^n}e^{-\sum x_i/\sigma}I_{(\mu,\infty)}(\min_i x_i)$$

Thus $T(\mathbf{X})=(\min_i X_i\,,\sum_{i=1}^n X_i)$ is sufficient for (μ,σ)

Exercise 6.4

$X_1,\dots,X_n$ are iid with pmf or pdf of the form

$$f(x\mid\boldsymbol{\theta})=h(x)c(\boldsymbol{\theta})\exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta})t_i(x)\right)$$

where $\boldsymbol{\theta}=(\theta_1,\theta_2,\dots,\theta_d),\quad d\le k$

The joint pmf or pdf may be written

$$f(\mathbf{x}\mid\boldsymbol{\theta})=\prod_{j=1}^n\left\{h(x_j)c(\boldsymbol{\theta})\exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta})t_i(x_j)\right)\right\}$$

$$=\underbrace{c(\boldsymbol{\theta})^n\exp\left(\sum_{i=1}^k w_i(\boldsymbol{\theta})\sum_{j=1}^n t_i(x_j)\right)}_{g(T(\mathbf{x})\mid\theta)}\underbrace{\prod_{j=1}^n h(x_j)}_{h(\mathbf{x})}$$

Thus $T(\mathbf{X})=\left(\sum_{j=1}^n t_1(X_j),\dots,\sum_{j=1}^n t_k(X_j)\right)$ is sufficient for $\boldsymbol{\theta}$

Exercise 6.5

$X_1,\dots,X_n$ are independent with pdfs

$$f_{X_i}(x_i\,|\,\theta)=\begin{cases}\dfrac{1}{2i\theta} & -i(\theta-1)<x_i<i(\theta+1)\\ 0 & \text{otherwise}\end{cases}$$

Note that for all x_i we may write

$$f_{X_i}(x_i\,|\,\theta)=\frac{1}{2i\theta}I_{(-i(\theta-1)\,,\,i(\theta+1))}(x_i)$$

$$=\frac{1}{2i\theta}I_{(-\infty\,,\,i(\theta+1))}(x_i)I_{(-i(\theta-1)\,,\,\infty)}(x_i)$$

$$=\frac{1}{2i\theta}I_{(-\infty\,,\,\theta+1)}(x_i/i)I_{(-(\theta-1)\,,\,\infty)}(x_i/i)$$

Hence the joint pdf may be written

$$f(\mathbf{x}\,|\,\theta)=\prod_{i=1}^{n}\left\{\frac{1}{2i\theta}I_{(-\infty\,,\,\theta+1)}(x_i/i)I_{(-(\theta-1)\,,\,\infty)}(x_i/i)\right\}$$

$$=\left(\frac{1}{2\theta}\right)^n\left(\prod_{i=1}^{n}\frac{1}{i}\right)\left(\prod_{i=1}^{n}I_{(-\infty\,,\,\theta+1)}(x_i/i)\right)\left(\prod_{i=1}^{n}I_{(-(\theta-1)\,,\,\infty)}(x_i/i)\right)$$

$$=\left(\frac{1}{2\theta}\right)^n\frac{1}{n!}I_{(-\infty\,,\,\theta+1)}(\max_i(x_i/i))I_{(-(\theta-1)\,,\,\infty)}(\min_i(x_i/i))$$

Thus

$$T(\mathbf{X})=\left(\min_i(X_i/i)\,,\,\max_i(X_i/i)\right)$$

is sufficient for θ

Exercise 6.6

$X_1,\dots,X_n$ are iid with pdf

$$f(x\,|\,\alpha,\beta)=\begin{cases}\dfrac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta} & \text{for } x>0\\ 0 & \text{otherwise}\end{cases}$$

We may write: $f(x\,|\,\alpha,\beta)=\dfrac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}I_{(0,\infty)}(x)$

Hence the joint pdf may be written

$$f(\mathbf{x}\,|\,\alpha,\beta)=\prod_{i=1}^{n}\frac{1}{\beta^\alpha\Gamma(\alpha)}x_i^{\alpha-1}e^{-x_i/\beta}I_{(0,\infty)}(x_i)$$

$$=\underbrace{\left(\frac{1}{\beta^\alpha\Gamma(\alpha)}\right)^n\left(\prod_{i=1}^{n}x_i\right)^{\alpha-1}e^{-\sum x_i/\beta}}_{g(T(\mathbf{x})\,|\,\theta)}\;\underbrace{I_{(0,\infty)}(\min_i x_i)}_{h(\mathbf{x})}$$

Thus $T(\mathbf{X})=\left(\prod_{i=1}^{n}X_i\,,\,\sum_{i=1}^{n}X_i\right)$ is sufficient for (α,β)

Exercise 6.7

$(X_1,Y_1),\dots,(X_n,Y_n)$ are independent with a pdf that is given by

$$f_{XY}(x,y\,|\,\theta_1,\theta_2,\theta_3,\theta_4)=\frac{1}{(\theta_2-\theta_1)(\theta_4-\theta_3)}$$

when $\theta_1<x<\theta_2$ and $\theta_3<y<\theta_4$ and by

$$f_{XY}(x,y\,|\,\theta_1,\theta_2,\theta_3,\theta_4)=0$$

otherwise

Thus we may write

$$f_{XY}(x,y\,|\,\theta_1,\theta_2,\theta_3,\theta_4)=\frac{1}{(\theta_2-\theta_1)(\theta_4-\theta_3)}I_{(\theta_1,\theta_2)}(x)I_{(\theta_3,\theta_4)}(y)$$

Hence the joint pdf may be written

$$f(\mathbf{x},\mathbf{y}\,|\,\alpha,\beta)=\prod_{i=1}^{n} f_{XY}(x_i,y_i\,|\,\theta_1,\theta_2,\theta_3,\theta_4)$$

$$=\prod_{i=1}^{n}\frac{1}{(\theta_2-\theta_1)(\theta_4-\theta_3)} I_{(\theta_1,\theta_2)}(x_i) I_{(\theta_3,\theta_4)}(y_i)$$

$$=\left(\frac{1}{(\theta_2-\theta_1)(\theta_4-\theta_3)}\right)^n I_{(\theta_1,\infty)}(\min x_i) I_{(-\infty,\theta_2)}(\max x_i) I_{(\theta_3,\infty)}(\min y_i) I_{(\theta-\infty,\theta_4)}(\max y_i)$$

Thus $T(\mathbf{X},\mathbf{Y})=\left(\min X_i, \max X_i, \min Y_i, \max Y_i\right)$ is sufficient for $(\theta_1,\theta_2,\theta_3,\theta_4)$

Exercise 6.8

$X_1,\ldots,X_n$ are iid with pdf $f(x-\theta)$

The joint pdf takes the form $f(\mathbf{x}\,|\,\theta)=\prod_{i=1}^{n} f(x_i-\theta)$

Hence

$$\frac{f(\mathbf{x}\,|\,\theta)}{f(\mathbf{y}\,|\,\theta)}=\frac{\prod_{i=1}^{n} f(x_i-\theta)}{\prod_{i=1}^{n} f(y_i-\theta)}=\frac{\prod_{i=1}^{n} f(x_{(i)}-\theta)}{\prod_{i=1}^{n} f(y_{(i)}-\theta)}$$

For a general f this does not depend on θ if and only if $\mathbf{y}$ and $\mathbf{x}$ have the same order statistics

Thus the order statistics are minimal sufficient (cf. Theorem 6.2.13)

Exercise 6.9.a

$X_1,\ldots,X_n$ are iid with pdf $f(x\,|\,\theta)=\frac{1}{\sqrt{2\pi}}e^{-(x-\theta)^2/2}$

The joint pdf takes the form

$$f(\mathbf{x}\,|\,\theta)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}}e^{-(x_i-\theta)^2/2}$$

$$=(2\pi)^{-n/2}\exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{n}x_i^2-2n\theta\bar{x}+n\theta^2\right]\right\}$$

Hence

$$\frac{f(\mathbf{x}\,|\,\theta)}{f(\mathbf{y}\,|\,\theta)}=\exp\left\{-\frac{1}{2}\left[\left(\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}y_i^2\right)\right]-2n\theta(\bar{x}-\bar{y})\right\}$$

This does not depend on θ if and only if $\bar{y}=\bar{x}$

Thus $\bar{X}$ is a minimal sufficient statistic for θ (cf. Theorem 6.2.13)

Exercise 6.9.b

$X_1,\ldots,X_n$ are iid with pdf $f(x\,|\,\theta)=\begin{cases} e^{-(x-\theta)} & \theta<x<\infty \\ 0 & \text{otherwise}\end{cases}$

The joint pdf takes the form

$$f(\mathbf{x}\,|\,\theta)=\prod_{i=1}^{n}e^{-(x_i-\theta)}I_{(\theta,\infty)}(x_i)\;=e^{n\theta}e^{-\sum x_i}\prod_{i=1}^{n}I_{(\theta,\infty)}(x_i)$$

$$=e^{n\theta}e^{-\sum x_i}I_{(\theta,\infty)}(\min_i x_i)$$

Hence

$$\frac{f(\mathbf{x}\,|\,\theta)}{f(\mathbf{y}\,|\,\theta)}=\frac{e^{-\sum x_i}I_{(\theta,\infty)}(\min_i x_i)}{e^{-\sum y_i}I_{(\theta,\infty)}(\min_i y_i)}$$

This does not depend on θ if and only if $\min y_i=\min x_i$

Thus $\min_i X_i$ is a minimal sufficient statistic for θ

Exercise 6.9.c

$X_1, ..., X_n$ are iid with pdf $f(x|\theta) = \dfrac{e^{-(x-\theta)}}{\left(1+e^{-(x-\theta)}\right)^2}$

The joint pdf takes the form

$$f(\mathbf{x}|\theta) = \prod_{i=1}^{n} \frac{e^{-(x_i-\theta)}}{\left(1+e^{-(x_i-\theta)}\right)^2} = \frac{e^{n\theta}e^{-\sum x_i}}{\prod_{i=1}^{n}\left(1+e^{-(x_i-\theta)}\right)^2}$$

Hence

$$\frac{f(\mathbf{x}|\theta)}{f(\mathbf{y}|\theta)} = \frac{e^{\sum y_i - \sum x_i} \prod_{i=1}^{n}\left(1+e^{-(y_i-\theta)}\right)^2}{\prod_{i=1}^{n}\left(1+e^{-(x_i-\theta)}\right)^2} = \frac{e^{\sum y_{(i)} - \sum x_{(i)}} \prod_{i=1}^{n}\left(1+e^{-(y_{(i)}-\theta)}\right)^2}{\prod_{i=1}^{n}\left(1+e^{-(x_{(i)}-\theta)}\right)^2}$$

This does not depend on θ if and only if $\mathbf{y}$ and $\mathbf{x}$ and have the same order statistics

Thus the order statistics are minimal sufficient