Solutions to exercises - Week 44

Consistency

- Exercise 10.1

Limiting and asymptotic variance

- Exercises 10.5 and 10.6

Best unbiased estimators and asymptotic relative efficiency

- Exercise 10.9

Exercise 10.1

Let $X_1,....,X_n$ be iid with pdf

$$f(x\,|\,\theta)=\frac{1}{2}(1+\theta x) \quad \text{for} \quad -1<x<1, \quad -1<\theta<1$$

There are a number of different consistent estimators. We will look at the moment estimator

Note that

$$\mathrm{E}_\theta X = \int_{-1}^{1} x\frac{1}{2}(1+\theta x)dx = \frac{1}{2}\int_{-1}^{1}(x+\theta x^2)dx = \frac{1}{2}\left[\frac{x^2}{2}+\theta\frac{x^3}{3}\right]_{-1}^{1} = \frac{\theta}{3}$$

$$\mathrm{E}_\theta X^2 = \int_{-1}^{1} x^2\frac{1}{2}(1+\theta x)dx = \frac{1}{2}\int_{-1}^{1}(x^2+\theta x^3)dx = \frac{1}{2}\left[\frac{x^3}{3}+\theta\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{3}$$

$$\mathrm{Var}_\theta X = \frac{1}{3}-\left(\frac{\theta}{3}\right)^2 = \frac{1}{3}\left(1-\frac{\theta^2}{3}\right)$$

The moment estimator is

$$\hat{\theta}=3\bar{X}_n$$

The estimator is unbiased and

$$\mathrm{Var}_\theta\hat{\theta}=9\mathrm{Var}_\theta\bar{X}_n=9\frac{1}{3n}\left(1-\frac{\theta^2}{3}\right)=\frac{1}{n}\left(3-\theta^2\right)\to 0$$

Hence the estimator is consistent (Thm 10.1.3)

Exercise 10.5

We have $X_1, X_2, ...$ iid $n(\mu,\sigma^2)$ and let $\bar{X}_n=\frac{1}{n}\sum_{i=1}^{n}X_i$

In example 5.5.25 we showed that

$$\sqrt{n}\left(\frac{1}{\bar{X}_n}-\frac{1}{\mu}\right)\to n\left(0,\frac{\sigma^2}{\mu^4}\right)$$

so the asymptotic variance is σ^2/μ^4

But this does not imply that the variance of $T_n=\sqrt{n}/\bar{X}_n$ converges to a finite limit, cf. example 10.1.8

We will here look closer at some details in the example

a) We have

$$\mathrm{E}T_n^2 = \int_{-\infty}^{\infty} \left(\frac{\sqrt{n}}{x}\right)^2 f_{\bar{X}_n}(x)dx = \int_{-\infty}^{\infty} \frac{n}{x^2} \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-n(x-\mu)^2/(2\sigma^2)} dx$$

$$\geq \frac{n\sqrt{n}}{\sqrt{2\pi}\,\sigma} \int_0^1 \frac{1}{x^2} e^{-n(x-\mu)^2/(2\sigma^2)} dx \quad \geq \frac{n\sqrt{n}}{\sqrt{2\pi}\,\sigma} K_n \int_0^1 \frac{1}{x^2} dx$$

where

$$K_n = \min_{0\leq x\leq 1} e^{-n(x-\mu)^2/(2\sigma^2)}$$

Now

$$\int_0^1 \frac{1}{x^2} dx = \infty$$

It follows that $\mathrm{E}T_n^2 = \infty$ for all *n* , so $\mathrm{Var}T_n$ does not exist for any *n*

b) We assume $\mu \neq 0$ and redefine T_n as

$$T_n = \begin{cases} \sqrt{n} / \bar{X}_n & \text{when } |\bar{X}_n| \geq \delta \\ 0 & \text{when } |\bar{X}_n| < \delta \end{cases}$$

Then

$$\mathrm{E}T_n^2 = \int_{-\infty}^{-\delta} \frac{n}{x^2} \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-n(x-\mu)^2/(2\sigma^2)} dx + \int_{\delta}^{\infty} \frac{n}{x^2} \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-n(x-\mu)^2/(2\sigma^2)} dx$$

$$= 2\int_{\delta}^{\infty} \frac{n}{x^2} \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} e^{-n(x-\mu)^2/(2\sigma^2)} dx \quad \leq 2\frac{n}{\delta^2} \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} \int_{\delta}^{\infty} e^{-n(x-\mu)^2/(2\sigma^2)} dx < \infty$$

and it follows that $\mathrm{Var}T_n < \infty$

c) We consider first the case where $\mu > 0$

Then (with $Z \sim n(0,1)$)

$$P(-\delta < \bar{X}_n < \delta) = P\left(\frac{\sqrt{n}(-\delta-\mu)}{\sigma} < \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} < \frac{\sqrt{n}(\delta-\mu)}{\sigma}\right)$$

$$\leq P\left(Z < \frac{\sqrt{n}(\delta-\mu)}{\sigma}\right) \to 0$$

A similar argument shows that $P(-\delta < \bar{X}_n < \delta) \to 0$ when $\mu < 0$

Exercise 10.6

We assume that

$$Y_n \,|\, W_n = w_n \sim n\left(0, w_n + (1-w_n)\sigma_n^2\right)$$

$$W_n \sim \mathrm{Bernoulli}(p_n)$$

a) Note that we have

$$\mathrm{E}\left(Y_n \,|\, W_n\right) = 0$$

$$\mathrm{Var}\left(Y_n \,|\, W_n\right) = W_n + (1-W_n)\sigma_n^2$$

Thus we obtain

$$\mathrm{E}Y_n = \mathrm{E}\left[\mathrm{E}\left(Y_n \,|\, W_n\right)\right] = \mathrm{E}[0] = 0$$

$$\mathrm{Var}Y_n = \mathrm{E}\left[\mathrm{Var}\left(Y_n \,|\, W_n\right)\right] + \mathrm{Var}\left[\mathrm{E}\left(Y_n \,|\, W_n\right)\right]$$

$$= \mathrm{E}\left[W_n + (1-W_n)\sigma_n^2\right] + \mathrm{Var}[0]$$

$$= p_n + (1-p_n)\sigma_n^2$$

b) Let $Z \sim n(0,1)$ and $X_n \sim n(0,\sigma_n^2)$. Then

$$P(Y_n < a) = \mathrm{E}\left[I(Y_n < a)\right] = \mathrm{E}\left[\mathrm{E}\left(I(Y_n < a) \mid W_n\right)\right]$$
$$= \mathrm{E}\left[P(Y_n < a \mid W_n)\right]$$
$$= p_n P(Y_n < a \mid W_n = 1) + (1-p_n) P(Y_n < a \mid W_n = 0)$$
$$= p_n P(Z < a) + (1-p_n) P(X_n < a)$$
$$= p_n P(Z < a) + (1-p_n) P(Z < a/\sigma_n)$$

If $p_n \to 1$ and $\sigma_n^2 \to \infty$ such that $(1-p_n)\sigma_n^2 \to \infty$, we have that

$P(Y_n < a) \to P(Z < a)$ so $Y_n \to n(0,1)$

But

$$\mathrm{Var} Y_n = p_n + (1-p_n)\sigma_n^2 \to \infty$$

Exercise 10.9

Let $X_1, X_2, \ldots, X_n$ be iid Poisson(λ)

$T = \sum_{i=1}^n X_i$ is a complete sufficient statistics for λ

(cf. Theorem 6.2.25)

a) The best unbiased estimator of

$$\varphi_0 = P(X = 0) = e^{-\lambda}$$

is

$$\hat{\varphi}_0 = \left(1 - \frac{1}{n}\right)^T$$

cf. additional exercise for week 42

b) By Theorems 7.3.17 and 7.3.23 the best unbiased estimator of $\varphi_1 = P(X = 1) = \lambda e^{-\lambda}$ is given by

$$\hat{\varphi}_1 = \mathrm{E}\left(W_1 \mid T\right)$$

where $W_1 = I\{X_1 = 1\}$

Now we have:

$$E(W_1 \mid T = t) = P\left(X_1 = 1 \mid \sum_{i=1}^n X_i = t\right)$$

$$= \frac{P\left((X_1 = 1) \cap \left(\sum_{i=1}^n X_i = t\right)\right)}{P\left(\sum_{i=1}^n X_i = t\right)} = \frac{P\left((X_1 = 1) \cap \left(\sum_{i=2}^n X_i = t-1\right)\right)}{P\left(\sum_{i=1}^n X_i = t\right)}$$

$$= \frac{P(X_1 = 1) P\left(\sum_{i=2}^n X_i = t-1\right)}{P\left(\sum_{i=1}^n X_i = t\right)} = \frac{\lambda e^{-\lambda} \dfrac{[(n-1)\lambda]^{t-1}}{(t-1)!} e^{-(n-1)\lambda}}{\dfrac{(n\lambda)^t}{t!} e^{-n\lambda}}$$

$$= \frac{t!}{(t-1)!} \frac{(n-1)^{t-1}}{n^t} = \frac{t}{n}\left(1 - \frac{1}{n}\right)^{t-1}$$

Thus the best unbiased estimator of $\varphi_1 = \lambda e^{-\lambda}$ is

$$\hat{\varphi}_1 = \frac{T}{n}\left(1 - \frac{1}{n}\right)^{T-1}$$

c) The ML estimator of λ is $\hat{\lambda} = \overline{X} = T / n$

Hence the estimators in a and b may be written

$$\hat{\varphi}_0 = \left(1 - \frac{1}{n}\right)^{n\hat{\lambda}} \quad \text{and} \quad \hat{\varphi}_1 = \hat{\lambda}\left(1 - \frac{1}{n}\right)^{n\hat{\lambda}-1}$$

Moreover, the ML estimators of $\varphi_0 = e^{-\lambda}$ and $\varphi_1 = \lambda\, e^{-\lambda}$ are given by $e^{-\hat{\lambda}}$ and $\hat{\lambda}\, e^{-\hat{\lambda}}$

When comparing the ML-estimator and the best unbiased estimator, we only consider estimation of $\varphi_0 = e^{-\lambda}$. The arguments for $\varphi_1 = \lambda\, e^{-\lambda}$ are similar

By example 10.1.17 we have that

$$\sqrt{n}\left(e^{-\hat{\lambda}} - e^{-\lambda}\right) \to n(0\,,\ \lambda e^{-2\lambda})$$

For the UMVUE we obtain by a Taylor expansion (remember that $(a^x)' = a^x \log a$)

$$\hat{\varphi}_0 = \left[\left(1 - \frac{1}{n}\right)^n\right]^{\hat{\lambda}}$$

$$\approx \left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} + \left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} \log\left(\left(1 - \frac{1}{n}\right)^n\right)\left(\hat{\lambda} - \lambda\right)$$

Thus we have

$$\sqrt{n}\,(\hat{\varphi}_0 - \varphi_0) = \sqrt{n}\left(\hat{\varphi}_0 - \left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda}\right) + \sqrt{n}\left(\left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} - e^{-\lambda}\right)$$

$$\approx \left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} \log\left(\left(1 - \frac{1}{n}\right)^n\right)\sqrt{n}\left(\hat{\lambda} - \lambda\right) + \sqrt{n}\left(\left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} - e^{-\lambda}\right)$$

Now

$$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \to n(0,\lambda) \qquad \text{(in distribution)}$$

$$\left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} \log\left(\left(1 - \frac{1}{n}\right)^n\right) \to e^{-\lambda}\log(e^{-1}) = -e^{-\lambda}$$

$$\sqrt{n}\left(\left[\left(1 - \frac{1}{n}\right)^n\right]^{\lambda} - e^{-\lambda}\right) \to 0 \qquad (??)$$

By Slutsky's theorem (Thm 5.5.17) it follows that

$$\sqrt{n}\,(\hat{\varphi}_0 - \varphi_0) \to n(0\,,\ \lambda e^{-2\lambda})$$

It follows that the asymptotic relative efficiency of the best unbiased estimator with respect to the MLE is equal to one (which is not surprising since we know that the MLE is asymptotically efficient)

In order to compare the two estimators one has to compare their properties for finite values of n (e.g. using simulations)

d) For the data example we have n = 15 and

$$T = 10 + 7 + 8 + ... + 3 + 5 = 104$$

$$\hat{\lambda} = \frac{104}{15} = 6.93$$

This gives

$$\hat{\varphi}_0 = \left(1 - \frac{1}{15}\right)^{104} = 0.000765$$

$$e^{-\hat{\lambda}} = \exp(-104/15) = 0.000975$$

$$\hat{\varphi}_1 = \frac{104}{15}\left(1 - \frac{1}{15}\right)^{103} = 0.00568$$

$$\hat{\lambda} e^{-\hat{\lambda}} = \frac{104}{15}\exp(-104/15) = 0.00676$$