STK4011 and STK9011
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Bivariate distributions

Covers (most of) the following material from chapter 4:

- Section 4.1: pages 144-147
- Section 4.2: pages 152-155
- Section 4.3: pages 158-160

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Joint, marginal and conditional distributions

Consider a continuous bivariate random vector (X, Y)

For any set $A \subset \mathcal{R}^2$ we have that

$$P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$$

where $f(x,y)$ is the joint pdf

Note that

$$\iint_{\mathcal{R}^2} f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

The joint cdf is given by

$$F(x,y) = P(X \le x, Y \le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(s,t)\,dt\,ds$$

Note that

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\,\partial y}$$

The marginal pdfs of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad -\infty < y < \infty$$

The conditional pdf of X given that $Y = y$ is defined by

$$f(x \mid y) = \frac{f(x,y)}{f_Y(y)}$$

for any y such that $f_Y(y) > 0$

The conditional pdf of Y given that $X = x$ is defined by

$$f(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

for any x such that $f_X(x) > 0$

Examples 4.1.12 and 4.2.4

Consider the joint pdf

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Note that

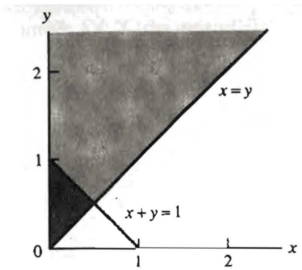
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{\infty}\int_0^{y} e^{-y}\,dx\,dy = \int_0^{\infty} y e^{-y} dy$$

$$= \int_0^{\infty} y^{2-1} e^{-y} dy = \Gamma(2) = 1\cdot\Gamma(1) = 1$$

We will calculate $P(X+Y \le 1) = P((X,Y)\in A)$

where $A = \{(x,y):\, x+y \le 1\}$

Hence



$$P(X+Y \le 1)$$
$$= P((X,Y)\in A)$$
$$= \iint_A f(x,y)\,dx\,dy$$
$$= \int_0^{1/2}\int_x^{1-x} e^{-y}\,dy\,dx$$
$$= \int_0^{1/2} (e^{-x} - e^{-(1-x)})\,dx = 1 + e^{-1} - 2e^{-1/2}$$

The marginal pdf of X is (when $x > 0$)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^{\infty} e^{-y} dy = e^{-x}$$

The conditional pdf of Y given that $X = x$ becomes (when $y > x > 0$)

$$f(y\,|\,x) = \frac{f(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{-(y-x)}$$

Independent random variables

Let (X, Y) be a bivariate random vector with joint pdf $f(x,y)$ and marginal pdfs $f_X(x)$ and $f_Y(y)$

X and Y are independent random variables if for every $x \in \mathcal{R}$ and $y \in \mathcal{R}$ we may write

$$f(x,y) = f_X(x)\, f_Y(y)$$

If X and Y are independent, we have

$$f(y\,|\,x) = \frac{f(x,y)}{f_X(x)} = \frac{f_X(x)\, f_Y(y)}{f_X(x)} = f_Y(y)$$

so knowledge that $X = x$ gives us no additional information about Y

Lemma 4.2.7

Let (X, Y) be a bivariate random vector with joint pdf $f(x,y)$

Then X and Y are independent random variables if and only it there exist non-negative functions $g(x)$ and $h(y)$ such that for every $x \in \mathcal{R}$ and $y \in \mathcal{R}$ we may write $f(x,y) = g(x)h(y)$

The usefulness of this result is that we do not need show that $g(x)$ and $h(y)$ integrate to one

Example 4.2.8

Consider the joint pdf

$$f(x,y) = \begin{cases} \frac{1}{384} x^2 y^4 e^{-y-x/2} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Define

$$g(x) = \begin{cases} x^2 e^{-x/2} & x>0 \\ 0 & x \le 0 \end{cases} \qquad h(y) = \begin{cases} y^4 e^{-y}/384 & y>0 \\ 0 & y \le 0 \end{cases}$$

Then $f(x,y) = g(x)h(y)$ so X and Y are independent

Expected values

Let (X, Y) be a bivariate random vector with joint pdf $f(x,y)$ and let $g(x,y)$ be a real valued function

Then the expected value of $g(X,Y)$ is given by

$$\mathrm{E}g(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy$$

provided that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x,y)| f(x,y)\, dx\, dy < \infty$

If X and Y are independent random variables and $g(x)$ is a function only of x and $h(y)$ is a function only of y, then

$$\mathrm{E}\big(g(X)h(Y)\big) = \big(\mathrm{E}g(X)\big)\big(\mathrm{E}h(Y)\big)$$

Moment generating function and distribution of the sum of independent random variables

If X and Y are independent random variables with moment generating functions $M_X(t) = \mathrm{E}e^{tX}$ and $M_Y(t) = \mathrm{E}e^{tY}$, then the moment generating function of $X+Y$ becomes

$$M_{X+Y}(t) = \mathrm{E}e^{t(X+Y)} = \mathrm{E}\left(e^{tX} e^{tY}\right)$$
$$= \left(\mathrm{E}e^{tX}\right)\left(\mathrm{E}e^{tY}\right) = M_X(t) M_Y(t)$$

For some important special cases we may use this result and the uniqueness of moment generating functions (cf. Theorem 2.3.11.b, page 65) to determine the distribution of $X+Y$

Example: sum of gamma variables

Assume that X and Y are independent random variables, and that

$$X \sim \text{gamma}(\alpha_1, \beta) \qquad Y \sim \text{gamma}(\alpha_2, \beta)$$

Then (for $t < 1/\beta$)

$$M_X(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_1} \qquad M_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_2}$$

Hence

$$M_{X+Y}(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha_1+\alpha_2}$$

and it follows that $X + Y \sim \text{gamma}(\alpha_1 + \alpha_2, \beta)$

The distribution of *U = g(X,Y)*

Let (X, Y) be a bivariate random vector with joint pdf $f(x, y)$ and let $g(x, y)$ be a real valued function

We want to find the pdf of $U = g(X,Y)$

One possibility is to first find the cdf of U and then differentiate to find the pdf

If we let $A_u = \{(x, y): g(x, y) \leq u\}$ the cdf is given by

$$F_U(u) = P(U \leq u) = P(g(X,Y) \leq u)$$

$$= P((X,Y) \in A_u) = \iint_{A_u} f(x, y)\,dx\,dy$$

Example

We have the joint pdf

$$f(x, y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Consider the random variable $U = X / Y$

The cdf is given by (when $0 < u < 1$)

$$F_U(u) = P(U \leq u) = P(X / Y \leq u) = P(X \leq uY)$$

$$= \int_0^\infty \int_0^{uy} e^{-y}\,dx\,dy = \int_0^\infty uye^{-y}dy = u\int_0^\infty ye^{-y}dy = u$$

By differentiating we find that the pdf is

$f_U(u) = 1 \quad \text{for} \quad 0 < u < 1$, so $U \sim \text{uniform}(0,1)$

Bivariate transformations

Let (X, Y) be a bivariate random vector with joint pdf $f_{X,Y}(x, y)$ and support

$$\mathcal{A} = \{(x, y) : f_{X,Y}(x, y) > 0\}$$

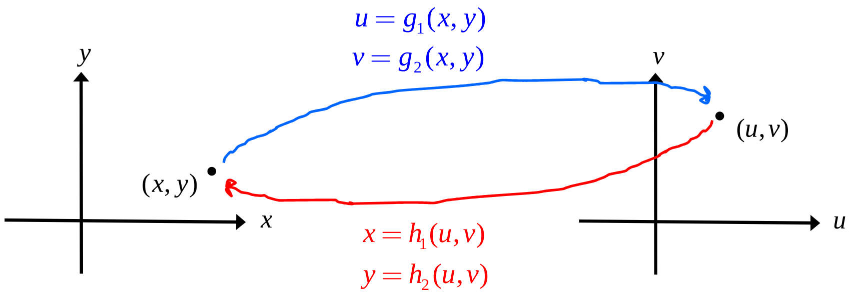
Let (U, V) be given by $U = g_1(X,Y)$ and $V = g_2(X,Y)$

Then the joint pdf $f_{U,V}(u,v)$ of (U, V) has support

$$\mathcal{B} = \{(u,v) : u = g_1(x, y) \text{ and } v = g_2(x, y) \text{ for some } (x, y) \in \mathcal{A}\}$$

We now assume that $u = g_1(x, y)$ and $v = g_2(x, y)$ defines a one-to-one transformation of $\mathcal{A}$ onto $\mathcal{B}$

We may then solve $u = g_1(x, y)$ and $v = g_2(x, y)$ to obtain the inverse transformation $x = h_1(u,v)$ and $y = h_2(u,v)$



Let $B \subset \mathcal{B}$ and define $A \subset \mathcal{A}$ by

$$A = \{(x,y) : x = h_1(u,v) \text{ and } y = h_2(u,v) \text{ for } (u,v) \in B\}$$

Then we have that $P\big((U,V) \in B\big) = P\big((X,Y) \in A\big)$

We also have

$$P\big((U,V) \in B\big) = \iint_B f_{U,V}(u,v)\,du\,dv \qquad (*)$$

Now by the formula for change of variables in a double integral we obtain

$$P((U,V) \in B) = P((X,Y) \in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_B f_{X,Y}(h_1(u,v),\, h_2(u,v))\,|J(u,v)|\,du\,dv \qquad (**)$$

where $J(u,v)$ is the Jacobian of the transformation:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Since (*) and (**) are valid for all $B \subset \mathcal{B}$, this shows that [for $(u,v) \in \mathcal{B}$]:

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v),\, h_2(u,v))\,|J(u,v)|$$

Example

Assume that X and Y are independent, and that

$X \sim \text{gamma}(\alpha_1, \beta)$ $\qquad Y \sim \text{gamma}(\alpha_2, \beta)$

Then the joint pdf of (X, Y) is given by

$$f_{X,Y}(x) = \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\, x^{\alpha_1-1} y^{\alpha_2-1} e^{-(x+y)/\beta} \quad \text{for } x, y > 0$$

Let (U, V) be given by $U = X + Y$ and $V = \dfrac{X}{X+Y}$

The joint pdf $f_{U,V}(u,v)$ of (U, V) has support

$$\mathcal{B} = \{(u,v) : u > 0 \text{ and } 0 < v < 1\}$$

The inverse transformation is given by $X = UV$ and $Y = U(1-V)$

The Jacobian is:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u(1-v) = -u$$

The joint pdf of (U, V) is given by $(u > 0,\ 0 < v < 1)$

$$f_{U,V}(u,v) = f_{X,Y}(uv,\, u(1-v))\,|-u|$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}\,(uv)^{\alpha_1-1}\,(u(1-v))^{\alpha_2-1}\,e^{-(uv+u(1-v))/\beta}\,u$$

$$= \frac{1}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\alpha_2)}\,u^{\alpha_1+\alpha_2-1}\,e^{-u/\beta}\,\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\,v^{\alpha_1-1}(1-v)^{\alpha_2-1}$$

Thus U and V are independent and

$U \sim \text{gamma}(\alpha_1+\alpha_2, \beta)$ $\qquad V \sim \text{beta}(\alpha_1, \alpha_2)$