STK4011 and STK9011
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Univariate distributions

Covers most of sections 3.2, 3.3 and 3.5
(and parts of sections 1.5,1.6, 2.2 and 2.3)

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Discrete distributions

A random variable X is discrete if its range $\mathcal{X} = \{x_1, x_2, \ldots.\}$ is countable (finite or countably infinite)

Cumulative distribution function (cdf)

$$F_X(x) = P(X \leq x)$$

is a step function

Probability mass function (pmf)

$$f_X(x) = P(X = x) = F_X(x) - F_X(x-) \quad \text{for } x \in \mathcal{X}$$

Note that $f_X(x) = 0$ for $x \notin \mathcal{X}$

The expected value or mean of $g(X)$ is given by

$$\mathrm{E}g(X) = \sum_{x \in \mathcal{X}} g(x) f_X(x)$$

provided that $\sum_{x \in \mathcal{X}} |g(x)| f_X(x) < \infty$

In particular

$$\mu = \mathrm{E}X = \sum_{x \in \mathcal{X}} x\, f_X(x)$$

$$\sigma^2 = \mathrm{Var}\, X = \mathrm{E}\{(X-\mu)^2\} = \mathrm{E}(X^2) - \mu^2$$

Moment generating function (mgf)

$$M_X(t) = \mathrm{E}\, e^{tX} = \sum_{x \in \mathcal{X}} e^{tx} f_X(x)$$

assuming that the expected value exists for all *t* in an open interval that contains zero

Note that $\mathrm{E}\, X^n = M_X^{(n)}(0)$

Hypergeometric distribution

We have an urn with N balls
M balls are red and $N-M$ balls are green
We select at random K balls
X is the number of red balls we select

Probability mass function (pmf)

$$f_X(x) = \frac{\binom{M}{x} \cdot \binom{N-M}{K-x}}{\binom{N}{K}} \qquad \begin{array}{l} x = 0,1,\ldots.,K \\ M-(N-K) \leq x \leq M \end{array}$$

$$\mathrm{E}X = K\frac{M}{N} \qquad \mathrm{Var}\, X = K\frac{M}{N}\left(1-\frac{M}{N}\right)\frac{N-K}{N-1}$$

Binomial distribution

We have n independent and identical Bernoulli trials with success probability p

X is the number of successes in the n trials

Probability mass function (pmf)

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0,1,\ldots,n$$

We have:

$$\mathrm{E}X = np \qquad \text{(example 2.2.3)}$$

$$\mathrm{Var}\, X = np(1-p) \qquad \text{(example 2.3.5)}$$

$$M_X(t) = \left[pe^t + (1-p)\right]^n \qquad \text{(example 2.3.9)}$$

Poisson distribution

X is Poisson distributed with parameter λ if its pmf takes the form

$$f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda} \qquad x = 0,1,\ldots$$

We have:

$$\mathrm{E}X = \mathrm{Var}\, X = \lambda$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

Relation with binomial distribution:

$$\binom{n}{x} p^x (1-p)^{n-x} \to \frac{\lambda^x}{x!} e^{-\lambda} \quad \text{when } n \to \infty,\ np \to \lambda$$

Negative binomial distribution

We have a sequence of independent Bernoulli trials with success probability p

X is the trial at which the r-th success occurs

Probability mass function (pmf)

$$f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad x = r, r+1, \ldots$$

Alternative formulation: $Y = X - r$ has pmf

$$f_Y(y) = \binom{r+y-1}{y} p^r (1-p)^y \qquad y = 0,1,\ldots$$

$$\mathrm{E}Y = \frac{r(1-p)}{p} = \mu \qquad \mathrm{Var}\, Y = \frac{r(1-p)}{p^2} = \mu + \frac{1}{r}\mu^2$$

Geometric distribution

We have a sequence of independent Bernoulli trials with success probability p

X is the trial at which the first success occurs

Probability mass function (pmf)

$$f_X(x) = p(1-p)^{x-1} \qquad x = 1,2,\ldots$$

$$\mathrm{E}X = \frac{1}{p}$$

$$\mathrm{Var}\, X = \frac{1-p}{p^2}$$

Alternatively we may consider $Y = X - 1$

Continuous distributions

If the cdf $F_X(x) = P(X \le x)$ is continuous, then X is a continuous random variable

For a continuous random variable (strictly speaking absolutely continuous random variable) we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

where $f_X(x)$ is the probability density function (pdf)

Further $f_X(x) = F_X'(x)$

The expected value or mean of $g(X)$ is given by

$$\mathrm{E}g(X) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

provided that $\int_{-\infty}^{\infty} |g(x)| f_X(x)\,dx < \infty$

In particular

$$\mu = \mathrm{E}X = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$\sigma^2 = \mathrm{Var}\,X = \mathrm{E}\{(X-\mu)^2\} = \mathrm{E}(X^2) - \mu^2$$

Note that if $\int_{-\infty}^{\infty} |x|^k f_X(x)\,dx < \infty$ for a *k* > 0, then

$\int_{-\infty}^{\infty} |x|^m f_X(x)\,dx < \infty$ for all *m* with 0 < *m* < *k*

Thus if $\mathrm{E}(X^k)$ exists, then $\mathrm{E}(X^m)$ exists for 0 < *m* < *k*

Moment generating function (mgf)

$$M_X(t) = \mathrm{E}\,e^{tX} = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

assuming that the expected value exists for all *t* in an open interval that contains zero

Note that $\mathrm{E}\,X^n = M_X^{(n)}(0)$

Normal distribution

If the pdf takes the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$$

then X is normally distributed with mean μ and variance σ^2

Short $X \sim n(\mu, \sigma^2)$

We have that

$$\mathrm{E}X = \mu$$

$$\mathrm{Var}\,X = \sigma^2$$

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

The variable

$$Z = \frac{X-\mu}{\sigma}$$

is standard normally distributed: $Z \sim n(0,1)$

To prove that the normal pdf integrates to 1, we have to prove that

$$\int_0^\infty e^{-z^2/2}\,dz = \sqrt{\frac{\pi}{2}}$$

or equivalently that

$$\left(\int_0^\infty e^{-z^2/2}\,dz\right)^2 = \frac{\pi}{2}$$

Location and scale families

Note that if

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

is the standard normal density, then the $n(\mu,\sigma^2)$ density may be written

$$f_X(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

This is an example of a location and scale family of distributions

In general we may obtain a location and scale family of distributions by starting with a standard pdf $f(z)$

The location and scale family is then given by all pdfs of the form

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

(one may easily check that these are pdfs)

μ is the location parameter $(-\infty < \mu < \infty)$

σ is the scale parameter $(\sigma > 0)$

The following result relates the location and scale family to the transformation of random variables (just as for the normal case)

Theorem 3.5.6
Let $f(\cdot)$ be any pdf. Let μ be any real number and let σ be any positive real number. Then X is a random variable with pdf $(1/\sigma)f((x-\mu)/\sigma)$ if and only if there exists a random variable Z with pdf $f(z)$ and $X = \sigma Z + \mu$

Note that if $\mathrm{E}Z$ and $\mathrm{Var}\,Z$ exist, then $\mathrm{E}X = \sigma \mathrm{E}Z + \mu$ and $\mathrm{Var}\,X = \sigma^2 \mathrm{Var}\,Z$

If $\sigma = 1$ we obtain a location family as all pdfs of the form

$$f(x-\mu)$$

If $\mu = 0$ we obtain a scale family as all pdfs of the form

$$\frac{1}{\sigma} f\left(\frac{x}{\sigma}\right)$$

Laplace distribution (double exponential distribution)

The standard Laplace distribution is given by the pdf

$$f(z) = \frac{1}{2} e^{-|z|} \qquad \text{for} \quad -\infty < z < \infty$$

If Z has this pdf, then $\mathrm{E}Z = 0$ and $\mathrm{Var}\, Z = 2$

Now $X = \sigma Z + \mu$ has pdf

$$f_X(x) = \frac{1}{2\sigma} e^{-|x-\mu|/\sigma} \qquad \text{for} \quad -\infty < x < \infty$$

Note that $\mathrm{E}X = \mu$ and $\mathrm{Var}\, X = 2\sigma^2$

Cauchy distribution

The standard Cauchy distribution is given by the pdf

$$f(z) = \frac{1}{\pi} \frac{1}{1+z^2} \qquad \text{for} \quad -\infty < z < \infty$$

One may show that $\mathrm{E}|Z| = \infty$ (example 2.2.4), so for the Cauchy distribution the mean does not exist

Now $X = Z + \theta$ has pdf

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2} \qquad \text{for} \quad -\infty < x < \infty$$

The mean and variance of X do not exist

Exponential distribution

The standard exponential pdf takes the form

$$f(z) = \begin{cases} e^{-z} & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

If Z has this pdf, then $\mathrm{E}Z = 1$ and $\mathrm{Var}\, Z = 1$

Then $X = \beta Z$ is exponentially distributed with scale parameter β :

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta} \qquad \text{for} \quad x > 0$$

We have $\mathrm{E}X = \beta$ and $\mathrm{Var}\, X = \beta^2$

Gamma distribution

The standard gamma pdf with shape parameter $\alpha > 0$ is given by

$$f(z) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z} & \text{if } z > 0 \\ 0 & \text{otherwise} \end{cases}$$

Here the gamma function $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz$ has the following properties:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$
$$\Gamma(n+1) = n! \quad \text{when } n \text{ is an integer}$$
$$\Gamma(1) = 1$$
$$\Gamma(1/2) = \sqrt{\pi}$$

Then $X = \beta Z$ is gamma distributed with shape parameter α and scale parameter β:

$$f_X(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad \text{for } x > 0$$

Short $X \sim gamma(\alpha, \beta)$

We have

$$\text{E}X = \alpha\beta \qquad \text{Var } X = \alpha\beta^2$$

$$M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha \quad \text{for } t < \frac{1}{\beta}$$

If $X \sim gamma(p/2, 2)$ we say that X is chi squared distributed with p degrees of freedom (p integer)

Lognormal distribution

If $\log X \sim n(\mu, \sigma^2)$, then X has a lognormal distribution

The pdf takes the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \frac{1}{x} e^{-(\log x - \mu)^2/(2\sigma^2)} \quad \text{for } x > 0$$

We have

$$\text{E}X = e^{\mu + \sigma^2/2}$$

$$\text{Var } X = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$$

Uniform distribution

If the pdf takes the form

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

then X is uniformly distributed over $[a, b]$

We have:

$$\text{E}X = \frac{b+a}{2}$$

$$\text{Var } X = \frac{(b-a)^2}{12}$$

Beta distribution

The beta distribution is a generalization of the uniform distribution on $[0\ ,\ 1]$

The $beta(\alpha,\beta)$ - distribution has pdf

$$f_X(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}(1-x)^{\beta-1} \quad \text{for } 0 < x < 1$$

Here the beta function is given by

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

We have

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

If $X \sim beta(\alpha,\beta)$ we have that

$$\mathrm{E}X^n = \frac{B(\alpha+n,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}$$

From this we obtain

$$\mathrm{E}X = \frac{\alpha}{\alpha+\beta}$$

$$\mathrm{Var}\,X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$