Supplementary exercises in STK4011/STK9011

Problem 1

Prove the identity (3.3.17) in Casella and Burger, i.e.

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Problem 2

Consider the exponential family where the pdf/pmf with the natural parameterization

$$f(x|\eta) = h(x)c^*(\eta)\exp(\Sigma_{i=1}^k \eta_i t_i(x))$$

a) Show that the natural parameter space is convex.

b) Show that $1/c(\eta)$ is a convex function.

Remember that a set A is *convex* if for all $x, y \in A$, $tx + (1-t)y \in A$ for all $t, 0 \leq t \leq 1$. A function $g(x)$ is *convex* if $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$ for all x, y and $0 < t < 1$.

Problem 3

Consider the exponential family where the pdf/pmf with the natural parameterization

$$f(x|\eta) = h(x)c^*(\eta)\exp(\Sigma_{i=1}^k \eta_i t_i(x))$$

Prove that, under the assumption that differentiation and integration can be interchanged,

a) $E[t_i(X)] = -\partial/\partial\eta_i \log(c^*(\eta)),\ i = 1, \cdots, k$

b) $Cov(t_i(X), t_j(X) = -\frac{\partial^2}{\partial\eta_i \partial/\eta_j} \log(c^*(\eta)),\ i, j = 1, \cdots, k$

Problem 4

Assume that the random variable X has pdf

$$f(x|\theta) = (x/\theta^2)\exp(-x^2/2\theta^2),\ x > 0, \theta > 0$$

which defines the *Rayleigh* distribution. Show that the distribution belongs to the exponential family of distributions and compute the expectation and variance of X^2.

Problem 5

Consider the exponential family where the pdf/pmf with the natural parameterization

$$f(x|\eta) = h(x)c^*(\theta)\exp(\Sigma_{i=1}^k \eta_i t_i(x))$$

Find an expression for the the moment generating function $E[\exp(\Sigma_{i=1}^k u_i t_i(x))]$, and verify that you get the same expectation and variance as in problem 3

Problem 6

Consider the gamma distribution with probability density function

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}\exp(-x/\beta),\ 0 < x < \infty,\ \alpha > 0, \beta > 0.$$

a) Assume that the parameter β is known. Write the pdf of this gamma distribution on the standard form and find the natural parameters η and the statistic T.

b) Assume that the parameter α is known. Write the pdf of this gamma distribution on the standard form and find the natural parameters η and the statistic T.

c) For the two gamma models discussed in part a) and b) express the expectations of $\log(X)$ by the natural parameter and vice versa.

Problem 7

The Pareto distribution is often used to model phenomena with heavy tails such as income. The density is

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta\alpha^\beta}{x^{\beta+1}} & \text{if } \alpha \leq x < \infty \quad \alpha,\beta > 0 \\ 0 & \text{else} \end{cases} \tag{1}$$

Assume that $X_1, \ldots, X_n$ is a random sample of random variables which are Pareto distributed so the probability density is $f(x|\alpha,\beta)$.

a) Show that the cumulative distribution function equals

$$F_X(x) = 1 - (\frac{\alpha}{x})^\beta \text{ for } x \geq \alpha.$$

b) Find $E[X^k]$ when X is Pareto distributed.

c) Show that the Pareto distribution belongs to the exponential family of distributions when α is a known constant. Also show that $\log X - \log \alpha$ has an exponential distribution with scale parameter $1/\beta$.

Problem 8

Let X and Y be independent, $X \sim U[0,1]$ and $Y \sim U[0,1]$. Find the pdf, probability density function, and mgf, moment generating function, of $X + Y$.

Problem 9

Let the variables $U_1, \dots, U_n$ be a random sample where $f(x) \sim \exp(-x),\ x > 0$. Let $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n-1)} \leq U_{(n)}$ be the order statistic

a) Let $n = 4$. Show that if $V_1 = U_{(1)}, V_2 = 3(U_{(2)} - U_{(1)}), V_3 = 2(U_{(3)} - U_{(2)}), V_4 = (U_{(4)} - U_{(3)})$, the simultanous density of (V_1, V_2, V_3, V_4) is $4\exp(-\beta(v_2 + v_3 + v_4) - 4v_1)$ for $0 < v_1, 0 < v_2, 0 < v_3, 0 < v_4$.

b) Show that for general n, the random variables $V_1 = U_{(1)}, V_i = (n-i+1)(U_{(i)} - U_{(i-1)}),,\ i = 2, \dots, n$ have simultaneous density $n\exp(-\beta(v_2 + \cdots + v_n) - n\beta v_1)$ for $0 < v_i,\ i = 1, \cdots, n$.

c) Why are $(V_1, \cdots, V_n)$ independent?

Problem 10

Consider a discrete sample $\mathbf{X}$ with probability mass function $f(\mathbf{x}|\theta)$. Show that if $T(\mathbf{X})$ is a sufficient and complete statistic and $U(\mathbf{X})$ is a minimal sufficient statistic, then $P(U(\mathbf{X}) = U(\mathbf{x})) = P(T(\mathbf{X}) = T(\mathbf{x}))$ for all $\mathbf{x}$ such that $P(\mathbf{X} = \mathbf{x}) > 0$.

Problem 11

Consider the situation where the random variables $X_1, \dots, X_n$ are independent and identically distributed with probability density function

$$f_X(x|\theta) = \begin{cases} \exp(-(x-\theta)) & \text{if } \theta \le x < \infty \\ 0 & \text{else} \end{cases},$$

where $-\infty < \theta < \infty$. Let $(X_{(1)}, \ldots, X_{(n)})$ be the order statistic.

a) Find the distribution of $X_{(1)} = \min X_i$ and show that it is sufficient.

b) Is it minimal sufficient?

c) Explain why $X_{(1)}$ is complete.

d) Explain why it follows from Basu's theorem that $X_{(1)}$ and $\Sigma_{i=1}^n (X_{(i)} - X_{(1)})$ are independent.

Problem 12

The Pareto distribution has probability density function

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta\alpha^\beta}{x^{\beta+1}} & \text{if } \alpha \le x < \infty \quad \alpha, \beta > 0 \\ 0 & \text{else} \end{cases} \tag{2}$$

Assume that $X_1, \ldots, X_n$ is a random sample of random variables which are Pareto distributed so the probability density is $f(x|\alpha,\beta)$.

a) Show that the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ of β and α are

$$\hat{\beta} = \frac{n}{\Sigma_{i=1}^n \log(X_i/X_{(1)})} \text{ and } \hat{\alpha} = X_{(1)}$$

where $X_{(1)} = \min_i X_i$.

b) Show that $\hat{\beta}$ and $\hat{\alpha}$ are independently distributed and that $2n\beta/\hat{\beta}$ is χ^2 distributed with $2(n-1)$ degrees of freedom and $\hat{\alpha}$ is Pareto distributed.

c) Show that $X_{(1)}$ and $\Sigma_{i=1}^n \log(X_i/X_{(1)})$ are sufficient statistics for α and β based on a random sample where the probability density is Pareto , i.e. has a probability distribution defined in equation (2).

It can also be shown the statistics $X_{(1)}$ and $\Sigma_{i=1}^n \log(X_i/X_{(1)})$ are complete statistics for α and β based on a random sample where the probability density is Pareto. Assuming that:

d) Find the UMVUE or best unbiased estimator for β.

e) Also show that $X_{(1)}[1 - \frac{1}{(n-1)}\frac{1}{\beta}]$ is the UMVUE or best unbiased estimator for α.

Problem 13

Assume that the observations $X_1, \dots, X_n$ are independent and identically distributed from a Poisson distribution with parameter λ. The probability mass function, pmf, is then

$$f_X(x|\lambda) = \frac{\lambda^x}{x!}e^{-\lambda},\ x = 0, 1, \dots$$

a) Use the delta method to find the asymptotic distribution of $\exp(-\bar{X})$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

b) Consider the reparameterization where $\gamma = \exp(-\lambda)$. Find the asymptic distribution of the maximum likelihood estimator, $\hat{\gamma}$, of γ using general results for the asymptotic distribution of maximum likelihood estimators.

c) Why is $\hat{\gamma} = \exp(-\bar{X})$? Verify that the asymptotic distributions from part a) and b) are identical.

Problem 14 (Schervish)

Let $(X, Y)'$ be binormal with expectation $(0, 0)'$ and covariance matrix $\begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}$, $|\theta| < 1$.

a) Find a two-dimensional minimal sufficient statistic.

b) Prove that the minimal sufficient statistic from part a) is not complete.

c) Prove that $Z_1 = X^2$ and $Z_2 = Y^2$ are both ancillary, but that (Z_1, Z_2) is not ancillary.

[Hint: Notice that $E[XY] = E[XE[Y|X]$ and use the properties of the conditional bivariate normal distribution.]

Problem 15 (Samuelsen, STK4011-f13)

Casella and Berger do not prove Slutsky's theorem: if the distribution of the random variable X_n

converges to the distribution of the random variable X and the random variable Y_n converges in probability to the scalar c, then (i) the distribution of X_n+Y_n converges toward the distribution of $X+c$ and (ii) the the distribution of X_nY_n converges toward the distribution of Xc.

a) Explain why it is sufficient to prove (i) for the case $c=0$.

b) Show that for all $\epsilon > 0$ $P(X_n + Y_n \leq x) \leq P(X_n \leq x + \epsilon) + P(|Y_n| > \epsilon)$.

c) Similarly, show that for all $\epsilon > 0$ $P(X_n \leq x - \epsilon) \leq P(X_n + Y_n \leq x) + P(|Y_n| > \epsilon)$.

d) Use the inequalities in parts c) and d) to prove (i) in Slutsky's theorem.

e) Prove (ii) in Slutsky's theorem.