UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4011/9011 — Statistical Inference Theory

Day of examination: Wednesday, December 8, 2021

Examination hours: 15:00 – 19:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator and "List of formulas for STK4011/9011 – Statistical Inference Theory (2021)".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that $X_1, \dots, X_n$ are independent and identically distributed $U(0, \theta)$ random variables with probability density function

$$f(x \mid \theta) = \begin{cases} 1/\theta, & \text{if } 0 < x < \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$.

(a) Find the probability density function of $X_{(n)} = \max_i(X_i)$.

(b) Show that both $\hat{\theta} = 2\bar{X}$ and $\tilde{\theta} = X_{(n)}(n+1)/n$ are unbiased estimators of θ.

(c) Show that $Var(\tilde{\theta}) \leq Var(\hat{\theta})$.

(d) Find a sufficient statistic for θ.

Problem 2

Assume that $X_1, \dots, X_n$ are independent and identically distributed Pareto(θ, ν) random variables with probability density function

$$f(x \mid \theta, \nu) = \begin{cases} \frac{\theta \nu^\theta}{x^{\theta+1}}, & \text{if } x \geq \nu, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$ and $\nu > 0$.

(a) Show that the maximum likelihood estimators of θ and ν are given by

$$\hat{\theta} = \frac{n}{T(\boldsymbol{X})} \quad \text{and} \quad \hat{\nu} = X_{(1)},$$

where $T(\boldsymbol{X}) = \log\left[(\prod_{i=1}^n X_i)/(X_{(1)})^n\right]$ and $X_{(1)} = \min_i(X_i)$.

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(b) Assuming that ν is unknown, show that the likelihood ratio test statistic for testing

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1$$

is given by

$$\lambda(\boldsymbol{X}) = \left(\frac{T(\boldsymbol{X})}{n}\right)^n e^{-T(\boldsymbol{X})+n}.$$

(c) Show that the likelihood ratio test is equivalent to rejecting H_0 if $T(\boldsymbol{X}) \leq c_1$ or $T(\boldsymbol{X}) \geq c_2$ for some constants $0 < c_1 < c_2$.

(d) Assuming you know the distribution of $T(\boldsymbol{X})$ under H_0, explain how you would choose c_1 and c_2 to obtain a likelihood ratio test of size α.

Problem 3

Assume that $X_1, \ldots, X_n$ is a random sample from an exponential population with probability density function

$$f(x \mid \lambda) = \lambda e^{-\lambda x}, \quad 0 \leq x < \infty,$$

where $\lambda > 0$. It can be shown that the maximum likelihood estimator is given by

$$\hat{\lambda}_n = \frac{n}{\sum_{i=1}^n X_i} = \frac{1}{\overline{X}_n}$$

and its expectation and variance are

$$E(\hat{\lambda}_n) = \frac{n}{n-1}\lambda \quad \text{and} \quad Var(\hat{\lambda}_n) = \frac{\lambda^2 n^2}{(n-1)^2(n-2)}.$$

(a) Show that $\sum_{i=1}^n X_i$ is a sufficient and complete statistic of λ.

(b) Find the best unbiased estimator of λ and compute its variance.

(c) Check if the best unbiased estimator from (b) attains the Cramer-Rao lower bound.

Now, for each $i = 1, \ldots, n$, define the random variables

$$Y_i = \begin{cases} 1, & \text{if } X_i > 1, \\ 0, & \text{otherwise,} \end{cases}$$

and define the following estimator of λ:

$$\tilde{\lambda}_n = -\log\left(\frac{\sum_{i=1}^n Y_i}{n}\right) = -\log(\overline{Y}_n).$$

Note that $Y_i \sim \text{Bernoulli}(p)$ with $p = P(X_i > 1)$.

(d) Show that $\tilde{\lambda}_n$ is a consistent estimator of λ.

(e) Show that $\sqrt{n}(\tilde{\lambda}_n - \lambda) \to N(0, e^\lambda - 1)$ in distribution as $n \to \infty$.

(f) Find the asymptotic relative efficiency (ARE) of $\tilde{\lambda}_n$ with respect to the maximum likelihood estimator $\hat{\lambda}_n$. Which estimator is better based on the asymptotic variance?

THE END - GOOD LUCK!