UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4011/9011 — Statistical Inference Theory

Day of examination: Friday, December 2, 2022

Examination hours: 15:00 – 19:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator and a clean version of the list of results titled "List of formulas for STK4011/9011 – Statistical Inference Theory (2022)".

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that $X_1, \ldots, X_n$ are independent and identically distributed Uniform(α, β) random variables, for which the probability density function (pdf) is given by

$$f(x \mid \alpha, \beta) = \frac{1}{\beta - \alpha}, \quad \alpha \le x \le \beta, \quad \alpha < \beta.$$

(a) Show that $T = (U, V)$, where $U = X_{(1)} = \min(X_i)$ and $V = X_{(n)} = \max(X_i)$, is a sufficient statistic for (α, β).

(b) Show that the joint pdf of (U, V) is given by

$$f_{U,V}(u, v \mid \alpha, \beta) = \frac{n(n-1)}{(\beta - \alpha)^n}(v - u)^{n-2}, \quad \alpha \le u < v \le \beta.$$

(c) Determine the joint pdf of (R, W), where $R = V - U$ is the range and $W = U$.

Problem 2

Assume that $X_1, \ldots, X_n$ are independent and identically distributed Gamma(α, β) random variables with pdf

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0, \quad \alpha, \beta > 0.$$

Furthermore, using the above parameterization, we have that:

$$E(X) = \frac{\alpha}{\beta}, \quad Var(X) = \frac{\alpha}{\beta^2}, \quad M_X(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha} \text{ for } t < \beta.$$

(Continued on page 2.)

(a) Show that $E(X^k) = \beta^{-k}\dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ for $k > -\alpha$.

(b) Show that $\sum_{i=1}^n X_i \sim \text{Gamma}(\alpha n, \beta)$.

For the remaining subproblems, assume that $\alpha = 2$, that is, $X_1, \ldots, X_n$ are independent and identically distributed Gamma$(2, \beta)$ random variables with pdf:

$$f(x|\beta) = \beta^2 x e^{-\beta x}, \quad x > 0, \quad \beta > 0.$$

(c) Show that $T = \sum_{i=1}^n X_i$ is a complete and sufficient statistic for β.

(d) Determine the distribution of T and, assuming $n \geq 2$, show that

$$E(T^{-1}) = \frac{\beta}{2n-1} \quad \text{and} \quad Var(T^{-1}) = \frac{\beta^2}{(2n-1)^2(2n-2)}.$$

Note: for the gamma function it holds that $\Gamma(z+1) = z\Gamma(z)$.

(e) Using the results in (d), find the best unbiased estimator of β and compute its variance.

(f) Does the best unbiased estimator in this case attain the Cramer-Rao lower bound?

Problem 3

Assume that $X_1, \ldots, X_n$ are independent and identically distributed Poisson(λ) random variables, for which the probability mass function (pmf) is given by

$$f(x \mid \lambda) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, 2, \ldots, \quad \lambda > 0.$$

Furthermore, we have that $E(X) = \lambda$ and $Var(X) = \lambda$.

(a) Show that $\hat{\lambda} = \frac{1}{n}\sum_{i=1}^n X_i$ is the maximum likelihood estimator of λ and explain why it is consistent.

(b) Derive the likelihood ratio test of $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$ and show that it is equivalent to rejecting H_0 if $\overline{X} \geq k$ for some $k \geq 0$.

(c) Is the test derived in (b) uniformly most powerful (UMP)? Motivate your answer. Here you may use the fact that $\sum_{i=1}^n X_i \sim \text{Poisson}(n\lambda)$.

(d) Derive the likelihood ratio test of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Assuming you do not know anything about the distribution of the test statistic, describe how the test can be used to find an approximate $1 - \alpha$ confidence set (or interval) for λ.

THE END - GOOD LUCK!