STK 4021 - Applied Bayesian Analysis and Numerical Methods
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Problem Set 1

Problem 1 (Jeffreys' prior). Jeffreys suggested a default rule for generating a prior distribution of a parameter θ in a sampling model $p(x|\theta)$. Jeffreys' prior is given by $p_J(\theta) \propto \sqrt{I(\theta)}$, where

$$I(\theta) = -\mathbb{E}\Big[\frac{\partial^2 \log p(X|\theta)}{\partial\theta^2}\Big|\theta\Big]$$

is the *Fisher information.*

(a) Let $X \sim$ binomial(n, θ). Obtain Jeffrey's prior distribution $p_J(\theta)$ for this model.

(b) Reparameterize the binomial sampling model with $\psi = \log\frac{\theta}{1-\theta}$, so that $p(x|\psi) = \binom{n}{x}e^{\psi x}(1+e^{\psi})^{-n}$. Obtain Jeffrey's prior distribution $p_J(\psi)$ for this model.

(c) Take the prior distribution from (a) and apply the change of variables formula to obtain the induced prior density on ψ. This density should be the same as the one derived in part (b) of this exercise. This consistency under reparameterization is the defining characteristic of Jeffreys' prior.

Problem 2 (improper Jeffreys' prior).

(a) Let $Y \sim$ Poisson(λ). Apply Jeffreys' procedure to this model. Does this produce an actual probability density for λ?

(b) Obtain the form of the function $f(\lambda, y) = \sqrt{I(\lambda)} \times p(y|\lambda)$. What probability density for λ is $f(\lambda, y)$ proportional to? Can we think of

$$\frac{f(\lambda, y)}{\int f(\lambda, y)dy}$$

as a posterior density of λ given $Y = y$?

Problem 3 (normal model with unknow mean and variance). Let $\mathcal{D} = \{X_1, \ldots, X_n\}$ be a sample with sampling model

$$p(x|\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(-\frac{(x-\theta)^2}{2\sigma^2}\Big)$$

and assume the following prior distributions

$$\theta|\sigma^2 \sim \mathcal{N}\Big(\mu_0, \frac{\sigma^2}{\kappa_0}\Big)$$
$$\frac{1}{\sigma^2} \sim \Gamma\Big(\frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\Big).$$

(a) Derive the formula for $p(\theta|\mathcal{D})$, the marginal posterior distribution of θ.

(b) Derive the formula for $p(\tilde{\sigma}^2|\mathcal{D})$, the marginal posterior distribution of the precision $\tilde{\sigma}^2 = 1/\sigma^2$.

(c) Check your work by comparing your formulas to Monte Carlo estimates of the marginal distributions, using some values of $\mathcal{D}, \mu_0, \sigma_0^2, \nu_0$, and κ_0 that you choose.

Solutions will be discussed in class on September 5.