Exercise 3, Ch. 11

Separate model

For each $j = 1, \dots, J$ $(J = 6)$:

$$y_{ij} \sim N(\theta_j, \sigma_j^2), \quad i = 1, \dots, n_j \quad (n_j = 5)$$

Non-inf. prior: $p(\theta_j, \sigma_j^2) \propto (\sigma_j^2)^{-1}$

} Ch. 3.2-model

Posterior distribution: $\left(y_{\cdot j} = \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij} \right)$

$$\sigma_j^2 | y \sim \text{Inv-}\chi^2\left(n_j - 1, \frac{1}{n_j - 1} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2\right)$$ (equiv to Inv-Gamma$\left(\frac{n_j-1}{2}, \frac{1}{2} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2\right)$)

$$\theta_j | y \sim t_{n_j - 1}\left(\bar{y}_{\cdot j}, \frac{1}{n_j(n_j-1)} \sum_{i=1}^{n_j} (y_{ij} - \bar{y}_{\cdot j})^2\right)$$

Corresponding posterior means and variances can then be found in App. A

(i) $\theta_6 | y \sim t_4\left(\bar{y}_{\cdot 6}, \frac{1}{5 \cdot 4} \cdot \sum_{i=1}^{5} (y_{i6} - \bar{y}_{\cdot 6})^2\right)$

(ii) Pred. of new measurement $\tilde{y}_6$ of machine 6: For $t = 1, \dots, T$: Draw $\theta_6^{(t)} | y$ and $\sigma_6^{(t)} | y$ and then $\tilde{y}_6^{(t)} | \theta_6^{(t)}, \sigma_6^{(t)}, y \sim N(\theta_6^{(t)}, \sigma_6^{2(t)})$

(iii) $\theta_7 | y$ is independent of the observed data, hence equal to the marginal prior for θ_7, which hence needs to be proper to be a true distribution. However, <u>with this model</u>, even with a proper prior distr., there is no info in the data about θ_7 (because each machine is modelled separately)

Pooled model

$$y_{ij} \sim N(\theta, \sigma^2), \quad i = 1, \dots, n_j, \; j = 1, \dots, J, \quad n = \sum_{j=1}^{J} n_j$$

Non-inf. prior: $p(\theta, \sigma^2) \sim (\sigma^2)^{-1}$

$$\sigma^2 | y \sim \text{Inv-Gamma}\left(\frac{n-1}{2}, \frac{1}{2} \sum_{j=1}^{J} \sum_{i=1}^{n_j} (y_{ij} - \bar{y})^2\right)$$

$$\theta | y \sim t_{n-1}\left(\bar{y}, \frac{1}{n(n-1)} \sum_{j=1}^{J} \sum_{i=1}^{n_j} (y_{ij} - \bar{y})^2\right)$$

i) $\theta | y$ given above $(\theta_6 = \theta)$

ii) For $t = 1, \dots, T$ sample $\sigma^{2(t)} | y$ and $\theta^{(t)} | y$, then $\tilde{y}_6 | \theta^{(t)}, \sigma^{2(t)}, y \sim N(\theta^{(t)}, \sigma^{2(t)})$

iii) $\theta_7 = \theta$ hence $\theta | y$ as given above

Hierarchical model

$y_{ij} \sim N(\theta_j, \sigma^2)$, $i=1,\dots,n_j$, $j=1,\dots,J$

$\theta_j \sim N(\mu, \tau^2)$, $j=1,\dots,J$

$p(\mu, \sigma^2, \tau^2) \propto \frac{1}{\sigma^2} \cdot \frac{1}{\tau}$

$\theta_j | \mu, \sigma, \tau, y \sim N(\theta_j^*, \sigma_j^{2*})$, where $\theta_j^* = \frac{1}{\frac{1}{\tau^2} + \frac{n_j}{\sigma^2}} \left(\frac{1}{\tau^2}\mu + \frac{n_j}{\sigma^2} \bar{y}_{\cdot j} \right)$

$\sigma_j^* = \frac{1}{\frac{1}{\tau^2} + \frac{n_j}{\sigma^2}}$

$\mu | \theta, \sigma, \tau, y \sim N(\mu^*, \tau^2/J)$, $\mu^* = \frac{1}{J}\sum_{j=1}^{J} \theta_j$

$\sigma^2 | \theta, \mu, \tau, y \sim \text{Inv-Gamma}\left(\frac{n}{2}, \frac{1}{2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{ij} - \bar{y})^2\right)$

$\tau^2 | \theta, \mu, \sigma, y \sim \text{Inv-Gamma}\left(\frac{J-1}{2}, \frac{1}{2}\sum_{j=1}^{J}(\theta_j - \mu)^2\right)$

(i) $\theta_6 | y$: Do Gibbs-sampling for all the parameters from the above full cond. distr. for $t=1,\dots,T$: Then $\theta_6^{(T/2+1)}, \dots, \theta_6^{(T)}$ approximates $\theta_6 | y$

(ii) From the Gibbs sampling samples: Draw $\tilde{y}_6^{(t)} \sim N(\theta_6^{(t)}, \sigma^{2(t)})$, then ...

(iii) — " ——————— : Draw $\tilde{\theta}_7^{(t)} \sim N(\mu^{(t)}, \tau^{2(t)})$, then ...

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