Exercise 3, Ch. 14

$$y \mid \beta, \sigma, X \sim N(X\beta, \sigma^2 I)$$
$$p(\beta, \sigma^2 \mid X) \propto \sigma^{-2}$$

$$p(\beta, \sigma^2 \mid y, X) \propto \frac{1}{\sigma^2} \cdot \frac{1}{\underbrace{|\sigma^2 I|^{1/2}}_{(\sigma^2)^{n/2}}} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right\}$$

$$p(\beta \mid \sigma, y, X) \propto \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right\} \qquad \propto \exp\left\{-\frac{1}{2\sigma^2}(\beta-\hat\beta)^T V_\beta^{-1}(\beta - \hat\beta)\right\}$$

$V_\beta = (X^T X)^{-1}$, $\hat\beta = (X^TX)^{-1}X^T y$

In order to show that $p(\beta \mid \sigma, y, X) = N(\beta \mid \hat\beta, \sigma^2 V_\beta)$, it is enough to show that

$$(y - X\beta)^T(y - X\beta) = (\beta - \hat\beta)^T V_\beta^{-1}(\beta - \hat\beta) + \text{const w.r.t. } \beta \qquad (AB)^T = B^T A^T$$

L.H.S: $(y - X\beta)^T(y - X\beta) = \underbrace{y^T y}_{\text{const. w.r.t } \beta} - y^T X\beta - \beta^T X^T y + \beta^T X^T X \beta$

R.H.S: $(\beta - \hat\beta)^T X^T X(\beta - \hat\beta) = \beta^T X^T X \beta - \beta^T X^T X \hat\beta - \hat\beta^T X^T X\beta + \underbrace{\hat\beta^T X^T X \hat\beta}_{\text{const. w.r.t. } \beta}$

$= \text{constant} + \beta^T X^T X \beta - \beta^T X^T X (X^T X)^{-1} X^T y - y^T X (X^T X)^{-1} X^T X \beta$

$= \text{Constant} + \beta^T X^T X \beta - \beta^T X^T y - y^T X\beta.$ Q.E.D.

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Exercise 4, Ch. 14

$$p(\beta, \sigma^2 \mid y) \propto (\sigma^2)^{-(\frac{n}{2}+1)} \exp\left\{-\frac{1}{2\sigma^2}(y - X\beta)^T(y - X\beta)\right\}$$

Bayes rule gives us

$$p(\sigma^2 \mid y) = \frac{p(\beta, \sigma^2 \mid y)}{p(\beta \mid \sigma^2, y)}$$

, which must be true for all values of β, also $\beta = \hat\beta$, which is convinient to work with

and $p(\beta \mid \sigma^2, y) = N(\beta \mid \hat\beta, \sigma^2 V_\beta)$

$$p(\sigma^2 \mid y) \propto \frac{(\sigma^2)^{-(\frac{n}{2}+1)} \cdot \exp\left\{-\frac{1}{2\sigma^2}(y - X\hat\beta)^T(y - X\hat\beta)\right\}}{|\sigma^2 V_\beta|^{-1/2} \exp\left\{\underbrace{-\frac{1}{2\sigma^2}(\hat\beta - \hat\beta)^T V_\beta^{-1}(\hat\beta - \hat\beta)}_{=0}\right\}}$$

Now $|\sigma^2 \underbrace{(X^T X)^{-1}}_{k \times k}| = (\sigma^2)^k \cdot \underbrace{|(X^T X)^{-1}|}_{\text{Constant w.r.t } \sigma^2}$

Hence $p(\sigma^2 \mid y) \propto (\sigma^2)^{-(\frac{n}{2}+1) + \frac{k}{2}} \cdot \exp\left\{-\frac{1}{2\sigma^2}(y - X\hat\beta)^T(y - X\hat\beta)\right\}$

$= (\sigma^2)^{-(\frac{n-k}{2}+1)} \exp\left\{-\frac{(n-k)}{2\sigma^2(n-k)}(y - X\hat\beta)^T(y - X\hat\beta)\right\}$

Hence $p(\sigma^2 \mid y) = \text{Inv-}\chi^2(n-k, s^2)$, $s^2 = \frac{1}{n-k}(y - X\hat\beta)^T(y - X\hat\beta)$

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Exercise 3, Ch. 16

Dose x_i	n_i	y_i
-0.86	5	0
-0.30	5	1
-0.05	5	3
0.73	5	5

Bioassay example $y_i \sim Bin(n_i, \theta_i)$

Before: $logit(\theta_i) = \alpha + \beta x_i \;, i = 1, \dots, 4$

$\theta_i = \frac{\exp\{z_i\}}{1+\exp\{z_i\}}$ $\quad k = 4$

Now: $logit(\theta_i) \sim N(\alpha + \beta x_i, \sigma^2)$

$\log\left(\frac{\theta_i}{1-\theta_i}\right)$

Hence $E[logit(\theta_i)] = \alpha + \beta x_i \;, i = 1, \dots, 4$

$Var[logit(\theta_i)] = \sigma^2$

Non-inf prior distr.: $p(\alpha, \beta, \sigma) \propto 1$ or $p(\alpha, \beta, \sigma^2) \propto \frac{1}{\sigma}$

Define $z_i = logit(\theta_i)$

$$p(z_i, \alpha, \beta, \sigma^2 | y) \propto \underbrace{\frac{1}{\sigma} \prod_{i=1}^{k} \frac{1}{\sigma}}_{\frac{1}{(\sigma^2)^{k/2+\frac{1}{2}}}} \exp\left\{-\frac{1}{2\sigma^2}\left(z_i - (\alpha + \beta x_i)\right)^2\right\} \cdot \prod_{i=1}^{k} \left(\frac{\exp\{z_i\}}{1+\exp\{z_i\}}\right)^{y_i} \left(\frac{1}{1+\exp\{z_i\}}\right)^{n_i - y_i}$$

$-(a+1) = -\frac{k+1}{2} \Rightarrow a = \frac{k-1}{2}$

$$p(\sigma^2 | \alpha, \beta, \theta, y) = \text{Inv-Gamma}\left(\frac{k-1}{2}, \frac{1}{2} \sum_{i=1}^{k} \left(z_i - (\alpha + \beta x_i)\right)^2\right)$$

$$p(\alpha, \beta | \theta, \sigma, y) \propto \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{k} \left(z_i - (\alpha + \beta x_i)\right)^2\right\}$$

Define $z = (z_1, \dots, z_k)^T$, $\gamma = (\alpha, \beta)^T$

$X = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_k \end{pmatrix}$

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Then $\sum_{i=1}^{k} \left(z_i - (\alpha + \beta x_i)\right)^2 = (z - X\gamma)^T (z - X\gamma)$

$$p(\alpha, \beta | \theta, \sigma, y) = N\left(\underbrace{\begin{pmatrix}\alpha \\ \beta\end{pmatrix}}_{\text{"}\gamma} \,\middle|\, \hat{\gamma}, V_\gamma \sigma^2\right), \quad \hat{\gamma} = (X^T X)^{-1} X^T z$$

$V_\gamma = (X^T X)^{-1}$

$$p(z_i | \alpha, \beta, \sigma, z_{(-i)}, y) \propto \left(\frac{e^{z_i}}{1 + e^{z_i}}\right)^{y_i} \left(\frac{1}{1 + e^{z_i}}\right)^{n_i - y_i} \exp\left\{-\frac{1}{2\sigma^2}\left(z_i - (\alpha + \beta x_i)\right)\right\}$$

Do Gibbs with Metropolis step for z_i, $i = 1, \dots, k$

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