Exercise 5, Ch. 5

Consider J parameters (vectors) $\theta_1, \dots, \theta_J$

The simplest form of an exchangeable prior distribution (appropriate if no information, other than the data, is available to distinguish)

$$p(\theta \mid \phi) = \prod_{j=1}^{J} p(\theta_j \mid \phi)$$, i.e. $\theta_1, \dots, \theta_J$ are modelled as conditionally independent given ϕ.

The unconditional prior for θ is found by

$$p(\theta) = \int \prod_{j=1}^{J} p(\theta_j \mid \phi)\, p(\phi)\, d\phi$$

Prove that $\mathrm{Cov}(\theta_i, \theta_j) \geq 0$ for all pairs (i, j)

We know that

$$\mathrm{Cov}(\theta_i, \theta_j) = E\big[\underbrace{\mathrm{Cov}(\theta_i, \theta_j) \mid \phi}_{=0}\big] + \mathrm{Cov}\big[E[\theta_i \mid \phi], \underbrace{E[\theta_j \mid \phi]}_{=E[\theta_i \mid \phi]}\big]$$

$$= \mathrm{Var}\big[E[\theta_i \mid \phi]\big] \geq 0 \qquad \text{Q.E.D.}$$

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Exercise 10, Ch. 5

We have the model

$$y_{ij} \mid \theta_j \sim N(\theta_j, \sigma^2), \text{ for } i = 1, \dots, n_j, \; j = 1, \dots, J, \quad \sigma^2 \text{ known}$$

$$p(\theta_1, \dots, \theta_J \mid \mu, \tau) = \prod_{j=1}^{J} N(\theta_j \mid \mu, \tau^2)$$

$$p(\mu, \tau) \propto p(\tau)$$

The joint posterior is

$$p(\theta, \mu, \tau \mid y) \propto p(\tau) \prod_{j=1}^{J} N(\theta_j \mid \mu, \tau^2) \cdot \prod_{j=1}^{J} \underbrace{\prod_{i=1}^{n_j} N(y_{ij} \mid \theta_j, \sigma^2)}_{\propto_{\text{w.r.t. } \theta_j} N(\bar{y}_{\cdot j} \mid \theta_j, \sigma_j^2)}$$

$$\bar{y}_{\cdot j} = \frac{1}{n_j} \sum_{i=1}^{n_j} y_{ij}, \qquad \sigma_j^2 = \frac{\sigma^2}{n_j}$$

Now

$$p(\mu, \tau \mid y) = \int p(\theta, \mu, \tau \mid y)\, d\theta = p(\tau) \prod_{j=1}^{J} \Big[\underbrace{\int N(\theta_j \mid \mu, \tau^2) \cdot N(\bar{y}_{\cdot j} \mid \theta_j, \sigma_j^2)\, d\theta_j}_{= N(\bar{y}_{\cdot j} \mid \mu, \sigma_j^2 + \tau^2)}\Big]$$

$$p(\mu \mid \tau, y) \underset{\text{w.r.t. } \mu}{\propto} p(\mu, \tau \mid y) \propto N(\mu \mid \mu^*, \sigma_\mu^{2*})$$

where $$\mu^* = \frac{\sum_{j=1}^{J} \frac{\bar{y}_{\cdot j}}{\sigma_j^2 + \tau^2}}{\sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2}}, \qquad \sigma_\mu^{2*} = \frac{1}{\sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2}}$$

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Bayes rule (reversed) gives us

$$p(\tau \mid y) = \frac{p(\mu, \tau \mid y)}{p(\mu \mid \tau, y)} \propto \frac{p(\tau) \cdot \prod_{j=1}^{J} N(\bar{y}_{\cdot j} \mid \mu^*, \sigma_j^2 + \tau^2)}{N(\mu^* \mid \mu^*, \sigma_\mu^{2*})}$$

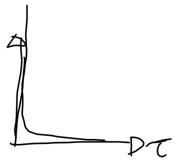
$$\propto p(\tau) \underbrace{\sigma_\mu^* \prod_{j=1}^{J} (\sigma_j^2 + \tau^2)^{-1/2} \exp\left\{ - \frac{(\bar{y}_{\cdot j} - \mu^*)^2}{2(\sigma_j^2 + \tau^2)} \right\}}_{(*)}$$

We want to inspect whether $p(\theta, \mu, \tau \mid y) = \underbrace{p(\tau \mid y)}_{?} \cdot \underbrace{p(\mu \mid \tau, y)}_{\text{Proper}} \cdot \underbrace{p(\theta \mid \mu, \tau, y)}_{\text{Proper}}$ is a proper distr. for choices of $p(\tau)$

a.) $p(\mu, \tau) \propto \tau^{-1}$ $\quad (\Leftrightarrow p(\mu, \log \tau) \propto 1)$

Let $\tau \to 0$: Then $(*) \to C > 0$ and $\frac{1}{\tau} \to \infty$

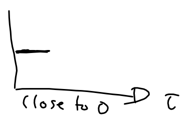
Hence $p(\tau \mid y)$ is not integrable!



b.) $p(\mu, \tau) \propto 1$: $\tau \to 0$: $(*) \to C$

$p(\tau) \propto 1$

OK



$$\tau \to \infty: \quad \exp\left\{ - \frac{(\bar{y}_{\cdot j} - \mu^*)^2}{2(\sigma_j^2 + \tau^2)} \right\} \leq 1$$

$$\sigma_\mu^* \cdot \prod_{j=1}^{J} (\sigma_j^2 + \tau^2)^{-1/2}$$

$$= \left(\sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2} \right)^{-1/2} \cdot \prod_{j=1}^{J} (\sigma_j^2 + \tau^2)^{-1/2}$$

$$= \left(\sum_{j=1}^{J} \prod_{i \neq j} (\sigma_i^2 + \tau^2) \right)^{-1/2} \leq \left(\sum_{j=1}^{J} \prod_{i \neq j} \tau^2 \right)^{-1/2} = \left(J \cdot (\tau^2)^{J-1} \right)^{-1/2} = J^{-1/2} \, \tau^{-(J-1)}$$

Hence, this is an upper bound for $(*)$, which is integrable for $J - 1 > 1 \Leftrightarrow J > 2$. Q.E.D.

Exercises 15, Chapter 5

A model

$$y_i \mid \theta_i \sim N(\theta_i, \sigma_i^2), \quad i = 1, \dots, J \quad (J = 227)$$

$$\theta_i \mid \mu, \tau^2 \sim N(\mu, \tau^2), \quad i = 1, \dots, J$$

$$p(\mu, \tau^2) \propto \frac{1}{\tau}$$

The joint posterior distr:

$$p(\theta, \mu, \tau^2 \mid y) \propto \frac{1}{\tau} \prod_{i=1}^{J} N(\theta_i \mid \mu, \tau^2) \cdot \prod_{i=1}^{J} N(y_i \mid \theta_i, \sigma_i^2)$$

gives the full conditional distributions

$$p(\tau^2 \mid \theta, \mu, y) \propto \text{Inv-Gamma}\left(\frac{J-1}{2}, \frac{1}{2}\sum_{i=1}^{J}(\theta_i - \mu)^2\right)$$

$$p(\mu \mid \theta, \tau^2, y) \propto N(\bar{\theta}, \tau^2/J)$$

$$p(\theta_i \mid \mu, \tau^2, y) \propto N\left(\frac{1}{\sigma_i^2 + \tau^2}(\sigma_i^2 \mu + \tau^2 y_i), \frac{\tau^2 \sigma_i^2}{\sigma_i^2 + \tau^2}\right), \quad i = 1, \dots, J$$

$$\frac{1}{\frac{1}{\sigma_i^2} + \frac{1}{\tau^2}}$$

b.)

$$E[\theta_i \mid \tau, y] = E\left[E[\theta_i \mid \mu, \tau, y] \mid \tau, y\right]$$

$$= E\left[\frac{\sigma_i^2 \mu + \tau^2 y_i}{\sigma_i^2 + \tau^2} \,\middle|\, \tau, y\right] = \frac{\sigma_i^2 \mu^* + \tau^2 y_i}{\tau^2 + \sigma_i^2}$$

$$Var[\theta_i \mid \tau, y] = E\left[Var[\theta_i \mid \tau, \mu, y] \mid \tau, y\right] + Var\left[E[\theta_i \mid \mu, \tau, y] \mid \tau, y\right]$$

$$= \frac{\tau^2 \sigma_i^2}{\sigma_i^2 + \tau^2} + \left(\frac{\sigma_i^2}{\tau^2 + \sigma_i^2}\right)^2 \cdot \sigma_\mu^{2*}$$