Exercises

Applied Bayesian Analysis and Numerical Methods (STK4021)

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Exercise 8

A random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\bar{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

- Prior $p(\theta) = N(\mu_0, \tau_0^2) = N(180, 40^2)$
- Sampling distribution $p(y|\theta) = N(\theta, \sigma^2) = N(\theta, 20^2)$

Exercise 8

(a) Give your posterior distribution for θ.

- $p(\theta|y) \propto p(y|\theta)p(\theta) = N(\theta|\mu_n, \tau_n^2)$ where:
- $$\mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} = \frac{\frac{1}{40^2}180 + \frac{n}{20^2}150}{\frac{1}{40^2} + \frac{n}{20^2}}$$
- $$\frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2} = \frac{1}{40^2} + \frac{n}{20^2}$$

Exercise 8

(b) A new student is sampled at random from the same population and has a weight of $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$.

- $p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)d\theta \propto N(\tilde{y}|\mu_n, \sigma^2 + \tau_n^2)$
- $p(\tilde{y}|y) \propto N(\tilde{y}|\frac{\frac{1}{40^2}180+\frac{n}{20^2}150}{\frac{1}{40^2}+\frac{n}{20^2}}, 20^2 + (\frac{1}{40^2} + \frac{n}{20^2})^{-1})$

Exercise 8

(c) For n = 10, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.

- $p(\theta|y) = N(\theta|\frac{\frac{1}{40^2}180+\frac{n}{20^2}150}{\frac{1}{40^2}+\frac{n}{20^2}}, (\frac{1}{40^2}+\frac{n}{20^2})^{-1})$
- $p(\theta|y) = N(\theta|150.73, 6.25^2)$

- $p(\tilde{y}|y) = N(\tilde{y}|\frac{\frac{1}{40^2}180+\frac{n}{20^2}150}{\frac{1}{40^2}+\frac{n}{20^2}}, 20^2+(\frac{1}{40^2}+\frac{n}{20^2})^{-1})$
- $p(\tilde{y}|y) = N(\tilde{y}|150.73, 20.95^2)$

Exercise 8

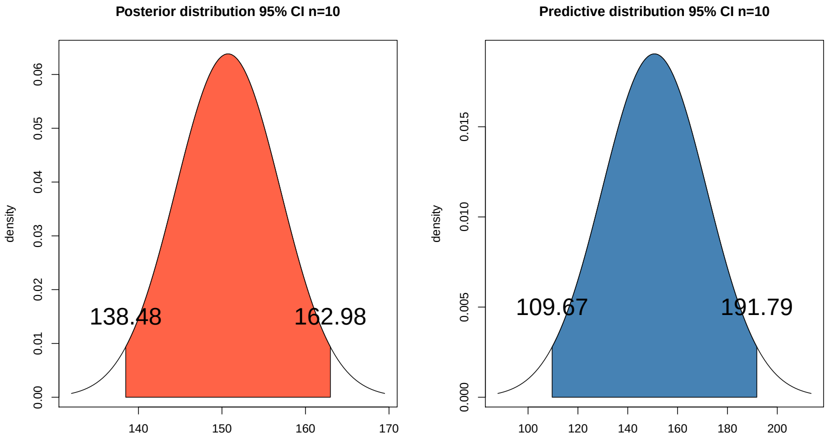

Figure: exercise 8 (c)

Exercise 8

(d) Do the same for n = 100.

- $p(\theta|y) = N(\theta|150.07, 1.997^2)$
- $p(\tilde{y}|y) = N(\tilde{y}|150.07, 20.10^2)$

Exercise 8

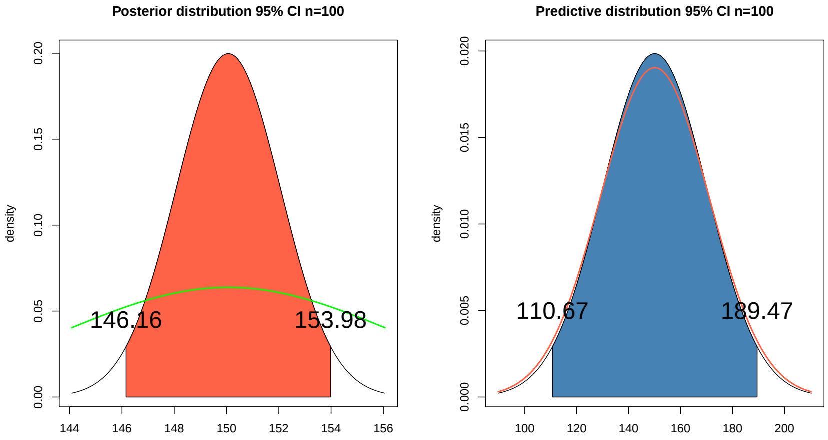

Figure: exercise 8 (d)

Exercise 8

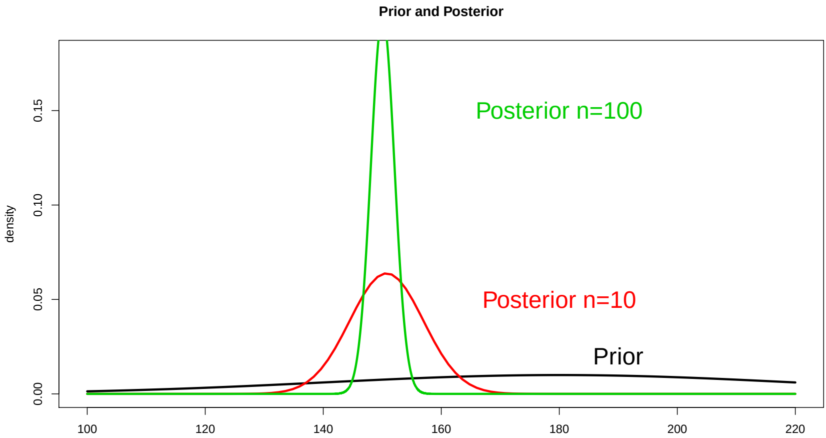

Figure: comparison

Exercise 9

Suppose your prior distribution for θ, the proportion of Californians who support the death penalty, is beta with mean 0.6 and standard deviation 0.3.

- (a) Determine the parameters α and β of your prior distribution. Sketch the prior density function.

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- $$\begin{cases} E(\theta) = \frac{\alpha}{\alpha+\beta} = 0.6 & (1) \\ Var(\theta) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = (0.3)^2 & (2) \end{cases}$$

- From (1) $\beta = \frac{2}{3}\alpha$, then plug in in (2)
- $p(\theta) = Beta(\theta|\alpha, \beta)$ with $\alpha = 1$ and $\beta = \frac{2}{3}$.

Exercise 9

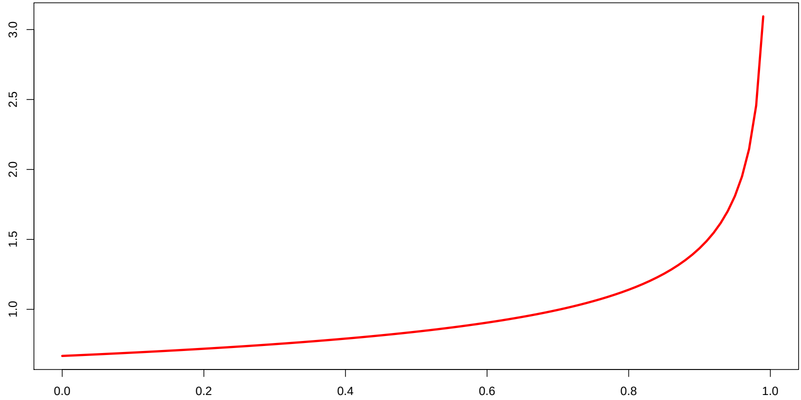

Figure: Prior

(b) A random sample of 1000 Californians is taken, and 65% support the death penalty. What are your posterior mean and variance for θ? Draw the posterior density function.

- Posterior distribution
  $p(\theta) = Beta(\theta|y + \alpha, n - y + \beta)$
- n = 1000 and y = 650
  $\Longrightarrow \; p(\theta) = Beta(651, 350.67)$

Exercise 9

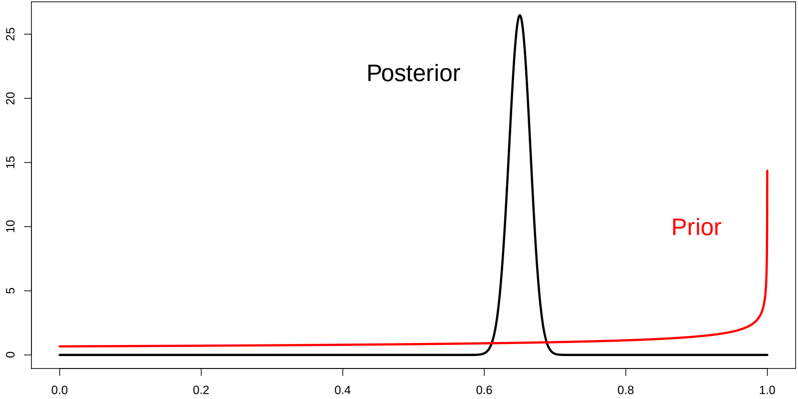

Figure: Posterior

(c) Examine the sensitivity of the posterior distribution to different prior means and widths including a non-informative prior.

Exercise 9

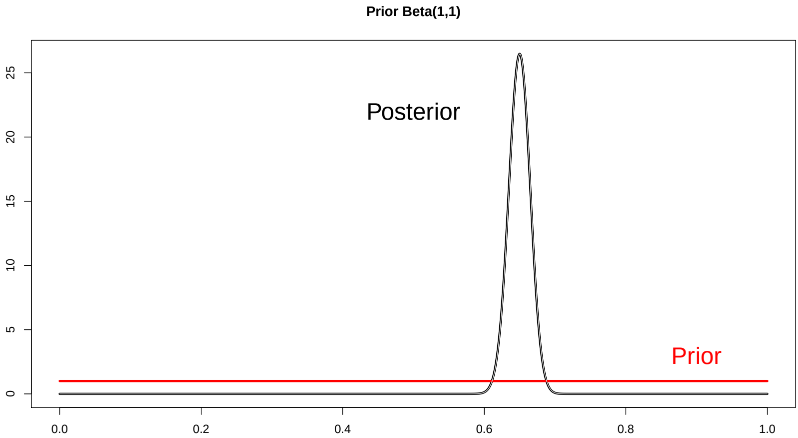

Figure: Exercise 9 (c)

Exercise 9

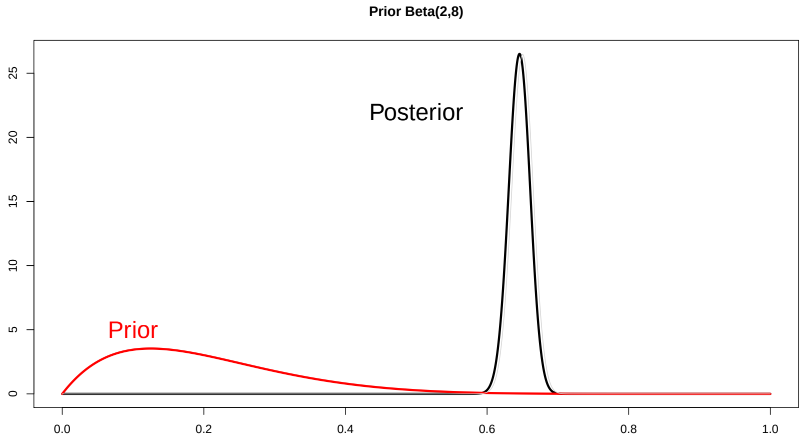

Figure: Exercise 9 (c)

Exercise 9

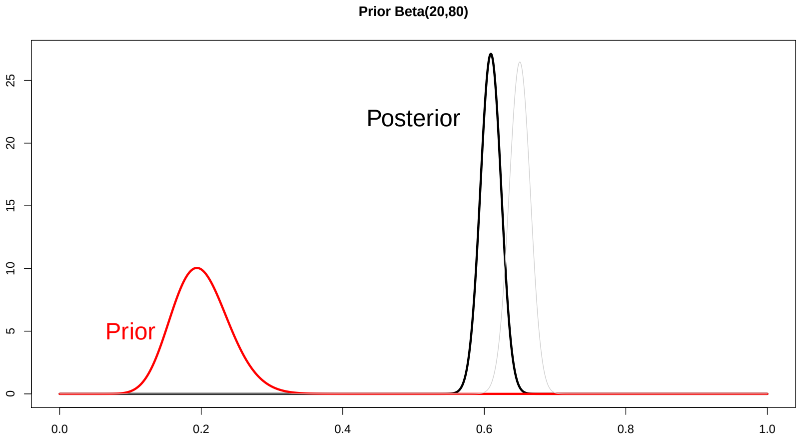

Figure: Exercise 9 (c)

Exercise 9

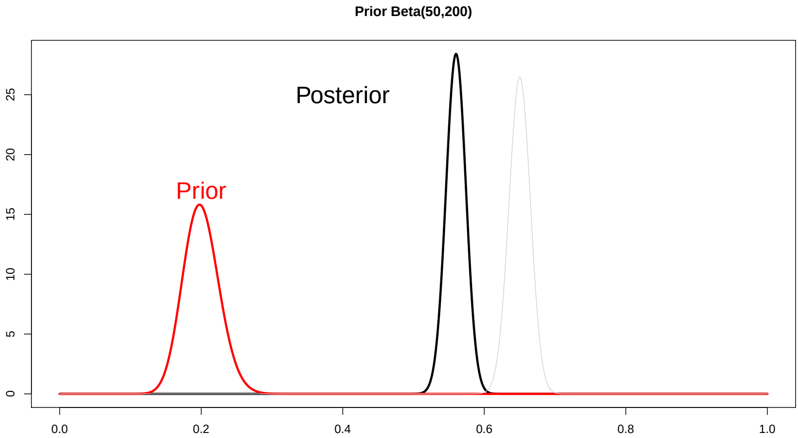

Figure: Exercise 9 (c)

Exercise 18

18. Poisson model: derive the gamma posterior distribution for the Poisson model parameterized in terms of rate and exposure with conjugate prior distribution.

Exercise 18

- Prior distribution
  $p(\theta) = Gamma(\theta|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$
- $p(y_i|\theta) \sim Poisson(x_i\theta)$ where θ = rate and x_i = exposure.
- $$p(y|\theta) = \prod_{i=1}^{n} \frac{(\theta x_i)^{y_i} e^{-\theta x_i}}{y_i!} =$$
- $$= \prod_{i=1}^{n} \frac{x_i^{y_i}}{y_i!}\theta^{y_i} e^{-\theta x_i} \propto \prod_{i=1}^{n} \theta^{y_i} e^{-\theta x_i}$$
- $$p(y|\theta) \propto \theta^{(\sum_{i=1}^{n} y_i)} e^{-(\theta \sum_{i=1}^{n} x_i)}$$

Exercise 18

- Posterior distribution $p(\theta|y) \propto p(y|\theta)p(\theta)$
- $$p(\theta|y) \propto \theta^{(\sum_{i=1}^{n} y_i)} e^{-(\theta \sum_{i=1}^{n} x_i)} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$
- $$p(\theta|y) \propto \theta^{(\sum_{i=1}^{n} y_i)} e^{-(\theta \sum_{i=1}^{n} x_i)} \theta^{\alpha-1} e^{-\beta\theta}$$
- $$p(\theta|y) \propto \theta^{(\alpha + \sum_{i=1}^{n} y_i - 1)} e^{-\theta(\beta + \sum_{i=1}^{n} x_i)}$$
- $$p(\theta|y) \sim Gamma(\alpha_n, \beta_n)$$
- $$Where \quad \alpha_n = \alpha + \sum^{n} y_i \quad and \quad \beta_n = \beta + \sum^{n} x_i$$

Exercise 12

Course Notes and Exercises by Nils Lid Hjort

12. Alarm or not?

Suppose y is binomial (n, θ), that the action space is $\{\text{alarm}, \text{no alarm}\}$, and that the loss function is as follows:

$$L(\theta, \text{no alarm}) = \begin{cases} 5000 & \text{if } \theta > 0.15, \\ 0 & \text{if } \theta < 0.15, \end{cases}$$
$$L(\theta, \text{alarm}) = \begin{cases} 0 & \text{if } \theta > 0.15, \\ 1000 & \text{if } \theta < 0.15. \end{cases}$$

Work out when the correct decision is 'alarm', in terms of the posterior distribution, having started with a given prior $p(\theta)$. In particular, for $n = 50$, for which values of y should one decide on 'alarm'? Sort out this for each of the following priors for θ.

(a) θ is uniform on $(0, 1)$.

(b) θ is a Beta $(2, 8)$.

(c) θ is an even mixture of a Beta $(2, 8)$ and a Beta $(8, 2)$.

Exercise 12

- Expected Loss

$$E[L(a|y)] = \int L(\theta, a) p(\theta|y) d\theta$$

- $E[L(alarm|y)] = 1000 \int_0^{0.15} p(\theta|y) d\theta$
- $E[L(no\ alarm|y)] = 5000 \int_{0.15}^{1} p(\theta|y) d\theta$

- (a) $p(\theta) \sim$Beta(1,1) $\Longrightarrow$ $p(\theta|y) \sim$ Beta(y+1,50-y+1)
- (b) $p(\theta) \sim$Beta(2,8) $\Longrightarrow$ $p(\theta|y) \sim$ Beta(y+2,50-y+8)

Exercise 12

- (c) $p(\theta) \sim \frac{1}{2}\text{Beta}(2{,}8)+\frac{1}{2}\text{Beta}(8{,}2)$
- $$p(\theta|y) \propto p(y|\theta)p(\theta) =$$
- $$= \theta^{y}(1-\theta)^{50-y} \cdot \frac{1}{2}[\theta^{1}(1-\theta)^{7} + \theta^{7}(1-\theta)^{1}] =$$
- $$= \frac{1}{2}[\theta^{y+2-1}(1-\theta)^{50-y+8-1}]+\frac{1}{2}[\theta^{y+8-1}(1-\theta)^{50-y+2-1}]$$
- $$p(\theta|y) \sim \frac{1}{2}Beta(y+2, 50-y+8)+\frac{1}{2}Beta(y+8, 50-y$$

Exercise 12

- $E[L(alarm|y)] = 1000 \int_0^{0.15} p(\theta|y)d\theta$
- $E[L(no\ alarm|y)] = 5000 \int_{0.15}^{1} p(\theta|y)d\theta$

- For which value of y:
  $E[L(alarm|y)] < E[L(no\ alarm|y)]$

Exercise 12

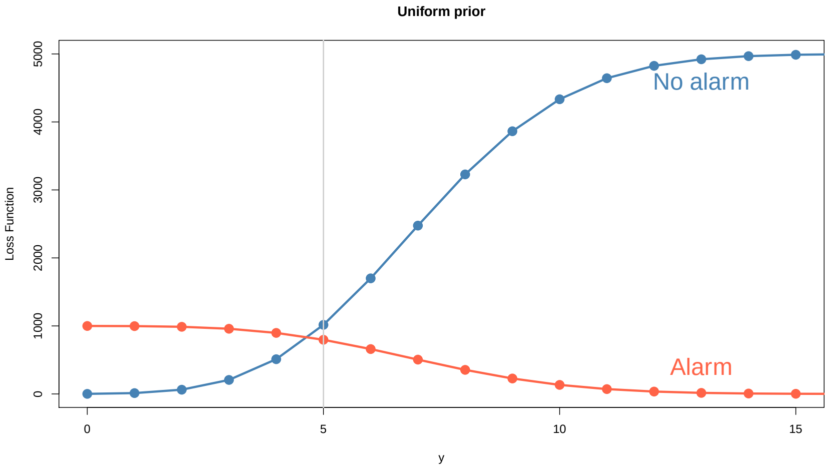

Figure: Exercise 12 (a)

Exercise 12

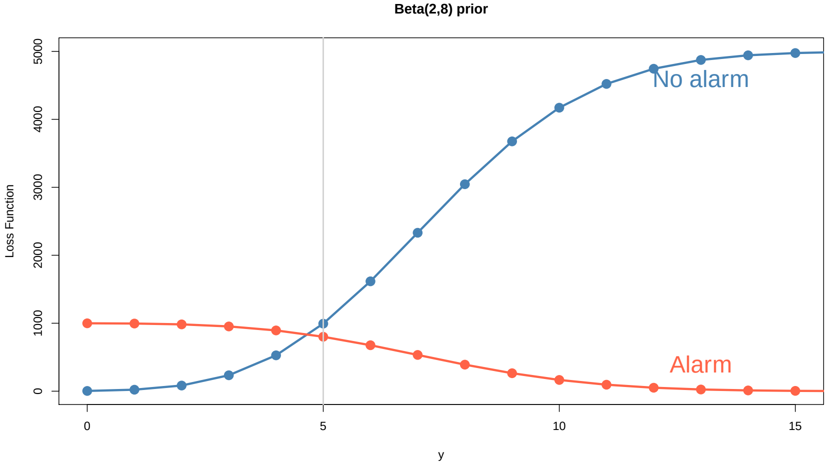

Figure: Exercise 12 (b)

Exercise 12

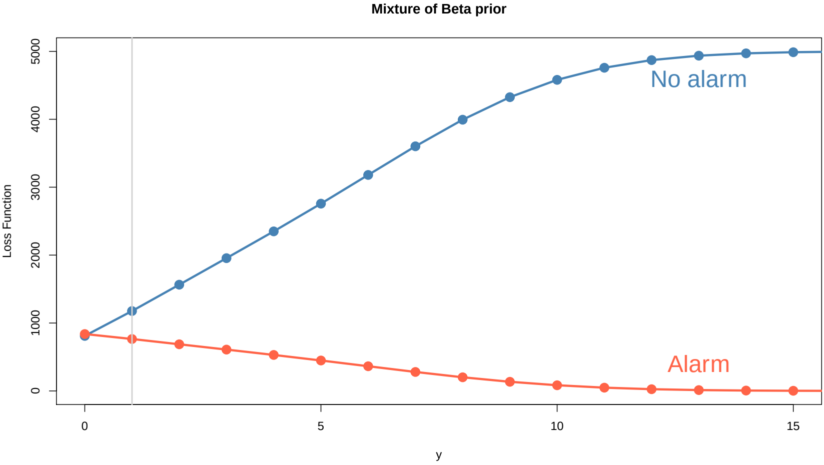

Figure: Exercise 12 (c)

Exercise 12

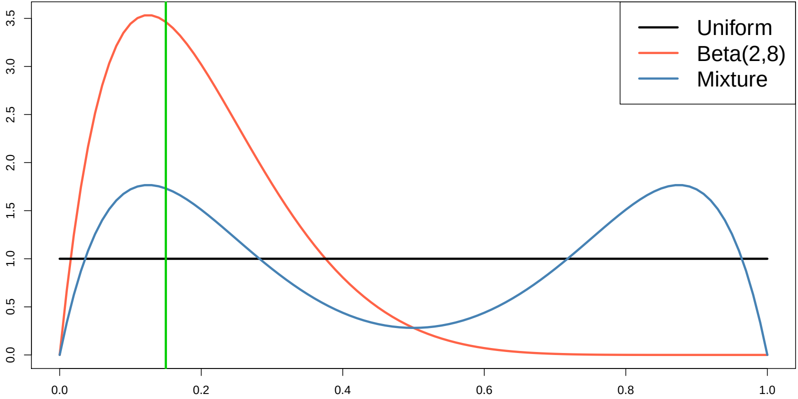

Figure: Exercise 12 Priors