STK4021 - Applied Bayesian Analysis

Exam 2023 Sample Solutions

Problem 1

(a) **True**, we only need the posterior mode

$$\theta_{\text{MAP}} = \arg\max_{\theta} \{\log \pi(\theta) + \log \pi(y \mid \theta)\}$$

and the Hessian

$$A = -\left.\frac{\partial^2}{\partial \theta^2} \{\log \pi(\theta) + \log \pi(y \mid \theta)\}\right|_{\theta=\theta_{\text{MAP}}}.$$

(b) **False**, de Finetti's theorem requires an *infinite* exchangeable sequence. A counterexample with $n = 2$ was given in lectures.

(c) **False**, the Metropolis-Hastings algorithm allows us to sample approximately from the posterior distribution, but does not yield an estimate of the marginal likelihood on its own.

Problem 2

(a) Clearly $f(y, \theta) \geq 0$ for all $y > 0, \theta > 0$. Also, making the substitution $u = \theta y/\sqrt{2}$, we have

$$\int_0^\infty f(y,\theta)\,\mathrm{d}y = \sqrt{\frac{2}{\pi}}\theta \int_0^\infty \exp\left\{-\frac{1}{2}\theta^2 y^2\right\} \mathrm{d}y = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-u^2)\,\mathrm{d}u = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1.$$

(b) Likelihood:

$$L_n(\theta) = \prod_{i=1}^n \sqrt{\frac{2}{\pi}}\theta \exp\left\{-\frac{1}{2}\theta^2 y_i^2\right\} = \left(\frac{2}{\pi}\right)^{n/2} \theta^n \exp\left\{-\frac{1}{2}\theta^2 n w_n\right\}.$$

Log-likelihood:

$$\ell_n(\theta) = \frac{n}{2}\log 2 - \frac{n}{2}\log \pi + n \log \theta - \frac{1}{2}\theta^2 n w_n.$$

Setting the derivative of this equal to zero, the MLE $\hat{\theta}$ satisfies

$$\frac{n}{\hat{\theta}} - \hat{\theta} n w_n = 0.$$

That is, $\widehat{\theta} = 1/\sqrt{w_n}$.

For the normal approximation, we use that

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\text{d}} Z \sim \text{N}(0, \mathcal{I}(\theta)^{-1})$$

under the true model, where in this case

$$\mathcal{I}(\theta) = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(y,\theta)\right] = \mathbb{E}\left[\frac{1}{\theta^2} + Y^2\right] = \frac{1}{\theta^2} + \text{Var}Y - (\mathbb{E}Y)^2 = \frac{2}{\theta^2}.$$

Hence our normal approximation is given by

$$\hat{\theta} \approx \text{N}\left(\theta, \frac{\theta^2}{2n}\right).$$

(c) The Jeffreys prior π_{J} satisfies

$$\pi_{\text{J}}(\theta) \propto \sqrt{1/\theta^2} = 1/\theta,$$

which is improper as $\int_0^\infty \mathrm{d}\theta/\theta$ diverges.

(d) We have

$$\pi(\theta \mid y_1, \ldots, y_n) \propto L_n(\theta)\pi(\theta) \propto \theta^{2\alpha+n-1}\exp\left\{-\theta^2\left(\frac{1}{2}nw_n + \frac{\alpha}{\beta}\right)\right\},$$

so by functional form, $\theta \mid y_1, \ldots, y_n \sim \text{Naka}(\alpha_n, \beta_n)$, where

$$2\alpha_n - 1 = 2\alpha + n - 1 \quad \text{and} \quad \frac{\alpha_n}{\beta_n} = \frac{\alpha}{\beta} + \frac{1}{2}nw_n.$$

That is,

$$\alpha_n = \alpha + n/2 \quad \text{and} \quad \beta_n = \frac{2\alpha + n}{2\alpha/\beta + nw_n}.$$

(e) Let

$$h(\theta) = \log\pi(\theta \mid y_1, \ldots, y_n) = (2\alpha_n - 1)\log\theta - \frac{\alpha_n}{\beta_n}\theta^2 + \text{constant}.$$

The maximiser θ_{MAP} satisfies $h'(\theta_{\text{MAP}}) = 0$, so

$$\frac{2\alpha_n - 1}{\theta_{\text{MAP}}} - 2\frac{\alpha_n}{\beta_n}\theta_{\text{MAP}} = 0.$$

That is,

$$\theta_{\text{MAP}} = \sqrt{\beta_n\left(1 - \frac{1}{2\alpha_n}\right)}.$$

Or, in terms of the prior parameters,

$$\theta_{\text{MAP}} = \sqrt{\frac{2\alpha + n}{2\alpha/\beta + nw_n}\,\frac{2\alpha + n - 1}{2\alpha + n}} = \sqrt{\frac{2\alpha + n - 1}{2\alpha/\beta + nw_n}}.$$

Also,

$$h''(\theta) = \frac{1 - 2\alpha_n}{\theta^2} - 2\frac{\alpha_n}{\beta_n},$$

so

$$A = -h''(\theta_{\text{MAP}}) = \frac{2\alpha_n - 1}{\beta_n\left(1 - \frac{1}{2\alpha_n}\right)} + 2\frac{\alpha_n}{\beta_n} = \frac{2\alpha_n}{\beta_n}\left\{\frac{2\alpha_n - 1}{2\alpha_n - 1} + 1\right\} = \frac{4\alpha_n}{\beta_n}.$$

Or, in terms of the prior parameters,

$$A = \frac{4\alpha}{\beta} + 2nw_n.$$

This yields

$$\theta \mid y_1, \ldots, y_n \approx \mathrm{N}\left(\sqrt{\beta_n\left(1 - \frac{1}{2\alpha_n}\right)}, \frac{\beta_n}{4\alpha_n}\right)$$

or

$$\theta \mid y_1, \ldots, y_n \approx \mathrm{N}\left(\sqrt{\frac{2\alpha + n - 1}{2\alpha/\beta + nw_n}}, \left\{\frac{4\alpha}{\beta} + 2nw_n\right\}^{-1}\right)$$

as the Laplace approximation.

Letting $z = \Phi^{-1}(0.975) \approx 1.96$, we get the credibility interval $\left[\theta_{\text{MAP}} \pm \frac{1}{2}\sqrt{\beta_n/\alpha_n}z\right]$.

(f) It suffices to show that if $\theta \sim \text{Naga}(\alpha, \beta)$, then

$$\mathbb{E}\theta = \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)}\sqrt{\frac{\beta}{\alpha}}.$$

Now, using the substitution $t = \alpha\theta^2/\beta$, we have

$$\begin{aligned}
\mathbb{E}\theta &= \frac{2}{\Gamma(\alpha)}\left(\frac{\alpha}{\beta}\right)^{\alpha}\int_0^\infty \theta^{2\alpha}\exp\left\{-\frac{\alpha}{\beta}\theta^2\right\}\mathrm{d}\theta \\
&= \frac{2}{2\Gamma(\alpha)}\left(\frac{\alpha}{\beta}\right)^{\alpha}\int_0^\infty \left(\frac{\beta}{\alpha}\right)^{\alpha+1/2} t^{\alpha-1/2}e^{-t}\,\mathrm{d}t \\
&= \frac{1}{\Gamma(\alpha)}\sqrt{\frac{\beta}{\alpha}}\int_0^\infty t^{(\alpha+1/2)-1}e^{-t}\,\mathrm{d}t \\
&= \frac{\Gamma(\alpha + 1/2)}{\Gamma(\alpha)}\sqrt{\frac{\beta}{\alpha}},
\end{aligned}$$

as required.

(g) We have

$$\begin{aligned}
\mathbb{P}(Y \le a) &= \mathbb{P}(-a \le X \le a) \\
&= \int_{-a}^{a}\frac{\theta}{\sqrt{2\pi}}\exp\left\{-\frac{\theta^2}{2}x^2\right\}\mathrm{d}x \\
&= \int_0^a \sqrt{\frac{2}{\pi}}\theta\exp\left\{-\frac{1}{2}\theta^2x^2\right\}\mathrm{d}x \\
&= \int_0^a \mathrm{HN}(x;\theta)\,\mathrm{d}x,
\end{aligned}$$

as required.

To sample from the half-normal distribution, we

1. sample $X \sim \mathrm{N}(0, 1/\theta^2)$,
2. set $Y \leftarrow |X|$.

Then $Y \sim \mathrm{HN}(\theta)$.

(h) The predictive is given by

$$\begin{aligned}\bar{f}(y) &= \int f(y,\theta)\pi(\theta \mid y_1,\ldots,y_n)\,\mathrm{d}\theta \\ &= \sqrt{\frac{2}{\pi}}\frac{2}{\Gamma(\alpha_n)}\left(\frac{\alpha_n}{\beta_n}\right)^{\alpha_n}\int_0^\infty \theta^{2\alpha_n}\exp\left\{-\left(\frac{\alpha_n}{\beta_n}+\frac{1}{2}y^2\right)\theta^2\right\}\,\mathrm{d}\theta.\end{aligned}$$

We recognise this integrand as an unnormalised Nakagami distribution, say $\mathrm{Naka}(\theta;\alpha',\beta')$, whose parameters satisfy

$$2\alpha' - 1 = 2\alpha_n \quad \text{and} \quad \frac{\alpha'}{\beta'} = \frac{\alpha_n}{\beta_n} + \frac{1}{2}y.$$

That is,

$$\alpha' = \alpha_n + 1/2 \quad \text{and} \quad \beta' = \frac{\alpha_n - 1/2}{\alpha_n/\beta_n + y/2}.$$

Hence we get

$$\begin{aligned}\bar{f}(y) &= \sqrt{\frac{2}{\pi}}\frac{2}{\Gamma(\alpha_n)}\left(\frac{\alpha_n}{\beta_n}\right)^{\alpha_n}\frac{\Gamma(\alpha_n+1/2)}{2}\left(\frac{\alpha_n}{\beta_n}+\frac{1}{2}y^2\right)^{-\alpha_n-1/2} \\ &= \sqrt{\frac{2}{\pi}}\frac{\Gamma(\alpha_n+1/2)}{\Gamma(\alpha_n)}\left(\frac{\alpha_n}{\beta_n}\right)^{\alpha_n}\left(\frac{\alpha_n}{\beta_n}\right)^{-\alpha_n-1/2}\left[1+\frac{\beta_n}{2\alpha_n}y^2\right]^{-\alpha_n-1/2} \\ &= 2\frac{\Gamma(\alpha_n+1/2)}{\Gamma(\alpha_n)}\left(\frac{\beta_n}{\pi\times 2\alpha_n}\right)^{1/2}\left[1+\frac{\beta_n}{2\alpha_n}y^2\right]^{-\alpha_n-1/2} \\ &= 2\,\mathrm{St}(y;0,\beta_n,2\alpha_n).\end{aligned}$$

The factor 2 comes from the fact that this is a half-Student's t-distribution, in much the same way as the half-normal distribution.

(i) Method 1:

1. sample $\theta \sim \mathrm{Naga}(\alpha_n, \beta_n)$,
2. sample $Y \mid \theta \sim \mathrm{HN}(\theta)$ (using the method given in part (g)).

Method 2:

1. sample $Z \sim \mathrm{St}(0, \beta_n, 2\alpha_n)$,
2. set $Y \leftarrow |Z|$.

Problem 3

(a) Noting that

$$\log|\beta^{-1}S|^{-1/2} = \log\left(\beta^{p/2}|S|^{-1/2}\right) = \frac{p}{2}\log\beta + \text{constant},$$

$$\begin{aligned}\log\pi(\mathbf{w},\beta) &= \frac{p}{2}\log\beta - \frac{\beta}{2}(\mathbf{w}-\mathbf{m})^\top S^{-1}(\mathbf{w}-\mathbf{m}) + (a-1)\log\beta - b\beta + \text{constant}\\ &= -\frac{\beta}{2}(\mathbf{w}-\mathbf{m})^\top S^{-1}(\mathbf{w}-\mathbf{m}) + \left(a+\frac{p}{2}-1\right)\log\beta - b\beta + \text{constant},\end{aligned}$$

as required.

(b) We have

$$\begin{aligned}\text{RHS} &= \frac{1}{2}(\mathbf{w}-S_n\mathbf{z})^\top S_n^{-1}(\mathbf{w}-S_n\mathbf{z}) - \frac{1}{2}\mathbf{z}^\top S_n\mathbf{z} + \frac{1}{2}\mathbf{m}^\top S^{-1}\mathbf{m} + \frac{1}{2}\mathbf{y}^\top\mathbf{y}\\ &= \frac{1}{2}\mathbf{w}^\top S_n^{-1}\mathbf{w} - \mathbf{z}^\top\mathbf{w} + \frac{1}{2}\mathbf{m}^\top S^{-1}\mathbf{m} + \frac{1}{2}\mathbf{y}^\top\mathbf{y}\\ &= \frac{1}{2}\mathbf{w}^\top S^{-1}\mathbf{w} + \frac{1}{2}\mathbf{w}^\top\Phi^\top\Phi\mathbf{w} - \mathbf{m}^\top S^{-1}\mathbf{w} - \mathbf{y}^\top\Phi\mathbf{w} + \frac{1}{2}\mathbf{m}^\top S^{-1}\mathbf{m} + \frac{1}{2}\mathbf{y}^\top\mathbf{y}\\ &= \frac{1}{2}(\mathbf{w}-\mathbf{m})^\top S^{-1}(\mathbf{w}-\mathbf{m}) + \frac{1}{2}(\mathbf{y}-\Phi\mathbf{w})^\top(\mathbf{y}-\Phi\mathbf{w})\\ &= \text{LHS}.\end{aligned}$$

(c) By the previous parts we have

$$\begin{aligned}\log\pi(\mathbf{w},\beta\mid\mathbf{y}) &= \log\pi(\mathbf{y}\mid\mathbf{w},\beta) + \log\pi(\mathbf{w},\beta) + \text{constant}\\ &= \frac{n}{2}\log\beta - \frac{\beta}{2}(\mathbf{y}-\Phi\mathbf{w})^\top(\mathbf{y}-\Phi\mathbf{w}) - \frac{\beta}{2}(\mathbf{w}-\mathbf{m})^\top S^{-1}(\mathbf{w}-\mathbf{m})\\ &\qquad + \left(a+\frac{p}{2}-1\right)\log\beta - b\beta + \text{constant}\\ &= -\frac{\beta}{2}(\mathbf{w}-S_n\mathbf{z})^\top S_n^{-1}(\mathbf{w}-S_n\mathbf{z}) + \frac{\beta}{2}\left\{\mathbf{z}^\top S_n\mathbf{z} - \mathbf{m}^\top S^{-1}\mathbf{m} - \mathbf{y}^\top\mathbf{y}\right\}\\ &\qquad + \left(a+\frac{p+n}{2}-1\right)\log\beta + b\beta + \text{constant}\\ &= -\frac{\beta}{2}(\mathbf{w}-S_n\mathbf{z})^\top S_n^{-1}(\mathbf{w}-S_n\mathbf{z}) + \left(a+\frac{p+n}{2}-1\right)\log\beta\\ &\qquad - \beta\left\{b - \frac{1}{2}\mathbf{z}^\top S_n\mathbf{z} + \frac{1}{2}\mathbf{m}^\top S^{-1}\mathbf{m} + \frac{1}{2}\mathbf{y}^\top\mathbf{y}\right\},\end{aligned}$$

so that

$$\mathbf{m}_n = S_n\mathbf{z}, \quad a_n = a + \frac{n}{2}, \quad b_n = b - \frac{1}{2}\mathbf{z}^\top S_n\mathbf{z} + \frac{1}{2}\mathbf{m}^\top S^{-1}\mathbf{m} + \frac{1}{2}\mathbf{y}^\top\mathbf{y}.$$

(d) Let

$$\begin{aligned}\mathcal{I}(\beta) &= \int \exp\left\{-\frac{\beta}{2}(\mathbf{w}-\mathbf{m}_n)^\top S_n^{-1}(\mathbf{w}-\mathbf{m}_n) - \frac{\beta}{2}(y' - \boldsymbol{\phi}(x')^\top\mathbf{w})^2\right\}\mathrm{d}\mathbf{w}\\ &= \beta^{-p/2}|T|^{-1}(2\pi)^{p/2}\exp\left\{-\frac{\beta}{2}\mathbf{m}_n^\top S_n^{-1}\mathbf{m}_n - \frac{\beta}{2}(y')^2 + \frac{\beta}{2}\mathbf{v}^\top T\mathbf{v}\right\}.\end{aligned}$$

Then

$$\begin{aligned}\pi(y' \mid \mathbf{y}) &= \int \pi(y' \mid \mathbf{w}, \beta)\pi(\mathbf{w}, \beta \mid \mathbf{y})\, \mathrm{d}\mathbf{w}\, \mathrm{d}\beta \\ &\propto \int \beta^{a_n+p/2-1/2} \exp\left\{-\beta b_n\right\} \mathcal{I}(\beta)\, \mathrm{d}\beta \\ &\propto \int \beta^{a_n-1/2} \exp\left\{-\beta\left(b_n + \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}\mathbf{m}_n + \frac{1}{2}(y')^2 - \frac{1}{2}\mathbf{v}^\top T\mathbf{v}\right)\right\} \mathrm{d}\beta.\end{aligned}$$

We recognise this integrand as an unnormalised Gamma(a', b') density, with

$$a' = a_n + 1/2, \quad b' = b_n + \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}\mathbf{m}_n + \frac{1}{2}(y')^2 - \frac{1}{2}\mathbf{v}^\top T\mathbf{v}.$$

Hence, integrating, we get

$$\pi(y' \mid \mathbf{y}) \propto \underbrace{\left[b_n + \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}\mathbf{m}_n + \frac{1}{2}(y')^2 - \frac{1}{2}\mathbf{v}^\top T\mathbf{v}\right]}_{\text{quadratic in } y'}^{-a_n-1/2}, \tag{1}$$

which by functional form is a Student's t-distribution with $\nu = 2a_n$.

For the sake of completeness, we also find the two other parameters. Let $\boldsymbol{\phi} = \boldsymbol{\phi}(x')$ for ease of notation. By the Sherman-Morrison formula we have

$$T = S_n - \frac{S_n\boldsymbol{\phi}\boldsymbol{\phi}^\top S_n}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}},$$

so that the quadratic $q(y')$ in (1) can be written as

$$q(y') = (y')^2\left[\frac{1}{2} - \frac{1}{2}\boldsymbol{\phi}^\top T\boldsymbol{\phi}\right] - (y')\left[\mathbf{m}_n^\top S_n^{-1}T\boldsymbol{\phi}\right] + c,$$

where

$$\begin{aligned}c &= b_n + \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}\mathbf{m}_n - \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}TS_n^{-1}\mathbf{m}_n \\ &= b_n + \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}\mathbf{m}_n - \frac{1}{2}\mathbf{m}_n^\top S_n^{-1}\left[S_n - \frac{S_n\boldsymbol{\phi}\boldsymbol{\phi}^\top S_n}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}\right]S_n^{-1}\mathbf{m}_n \\ &= b_n + \frac{1}{2}\frac{(\mathbf{m}_n^\top\boldsymbol{\phi})^2}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}.\end{aligned}$$

Also,

$$\mathbf{m}_n^\top S_n^{-1}T\boldsymbol{\phi} = \mathbf{m}_n^\top\boldsymbol{\phi} - \frac{\mathbf{m}_n^\top\boldsymbol{\phi}\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}} = \mathbf{m}_n^\top\boldsymbol{\phi}\left[1 - \frac{\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}\right] = \frac{\mathbf{m}_n^\top\boldsymbol{\phi}}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}$$

and

$$\frac{1}{2} - \frac{1}{2}\boldsymbol{\phi}^\top T\boldsymbol{\phi} = \frac{1}{2} - \frac{1}{2}\boldsymbol{\phi}^\top S_n\boldsymbol{\phi} + \frac{1}{2}\frac{(\boldsymbol{\phi}^\top S_n\boldsymbol{\phi})^2}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}} = \frac{1}{2} - \frac{1}{2}\frac{\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}} = \frac{1}{2}\frac{1}{1 + \boldsymbol{\phi}^\top S_n\boldsymbol{\phi}},$$

so that

$$
\begin{aligned}
q(y') &= \frac{1}{2}\frac{1}{1+\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}(y')^2 - \frac{\mathbf{m}_n^\top\boldsymbol{\phi}}{1+\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}(y') + \frac{1}{2}\frac{(\mathbf{m}_n^\top\boldsymbol{\phi})^2}{1+\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}} + b_n \\
&= \frac{1}{2}\frac{1}{1+\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}}(y' - \mathbf{m}_n^\top\boldsymbol{\phi})^2 + b_n \\
&\propto 1 + \frac{\frac{a_n}{b_n}\left[1+\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}\right]^{-1}\left(y' - \mathbf{m}_n^\top\boldsymbol{\phi}\right)^2}{2a_n},
\end{aligned}
$$

which forces

$$
\mu = \mathbf{m}_n^\top\boldsymbol{\phi}, \quad \lambda = \frac{a_n}{b_n}\left[1+\boldsymbol{\phi}^\top S_n\boldsymbol{\phi}\right]^{-1}.
$$

(e) By definition,

$$
\pi(\mathbf{y}) = \frac{\pi(\mathbf{w},\beta)\pi(\mathbf{y}\mid\mathbf{w},\beta)}{\pi(\mathbf{w},\beta\mid\mathbf{y})}.
$$

Since the prior is conjugate, we only have to work out the ratio of normalisation constants. Hence

$$
\pi(\mathbf{y}) = \frac{|S|^{-1/2}(2\pi)^{-p/2}b^a/\Gamma(a)\times(2\pi)^{-n/2}}{|S_n|^{-1/2}(2\pi)^{-p/2}b_n^{a_n}/\Gamma(a_n)} = \frac{1}{(2\pi)^{n/2}}\frac{b^a}{b_n^{a_n}}\frac{\Gamma(a_n)}{\Gamma(a)}\frac{|S_n|^{1/2}}{|S|^{1/2}}.
$$

Problem 4

(a) We have

$$
\begin{aligned}
x' &= \exp\left\{\log x + \varepsilon\right\}, \\
\varepsilon &= \log x' - \log x,
\end{aligned}
$$

so that

$$
q(x'\mid x) = q(\varepsilon(x'))\left|\frac{\partial\varepsilon}{\partial x'}\right| = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}\left\{\log x' - \log x\right\}^2\right\}\frac{1}{x'}.
$$

(b) Noting that the exponent from (a) is symmetric in x and x', we get

$$
\alpha(x'\mid x) = \min\left\{1, \frac{\pi(x')q(x\mid x')}{\pi(x)q(x'\mid x)}\right\} = \min\left\{1, \frac{\pi(x')}{\pi(x)}\frac{1/x}{1/x'}\right\} = \min\left\{1, \frac{\pi(x')x'}{\pi(x)x}\right\},
$$

as required.

(c) For example:

- Check that multiple runs from different starting points yield the same results, to the required accuracy.
- Draw trace-plots to inspect the effect of burn-in and autocorrelation.
- Plot autocorrelation against lag. Make sure it drops off quickly.
- Compute the effective sample size (ESS). Check that this is sufficiently large (> 1000, say).

- Calculate the acceptance rate. Make sure this close to the optimal value (about 24%).

(d) Writing

$$\hat{\mu} = \frac{1}{S}\sum_{s=1}^{S} X_s,$$

then we can use

$$\hat{\sigma}^2 = \frac{1}{S}\sum_{s=1}^{S}(X_s - \hat{\mu})^2 \quad \text{or} \quad \hat{\sigma}^2 = \frac{1}{S-1}\sum_{s=1}^{S}(X_s - \hat{\mu})^2.$$