UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4021 — Applied Bayesian statistics - Home exam

Day of examination: 2013

Examination hours: 15.00 – 19.00.

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Calculator, plus one single sheet of paper with the candidate's

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

(a) $y \sim N(\theta_0, \sigma_0^2 + 1)$

(b) We have

$$\begin{pmatrix}\theta\\ y\end{pmatrix} \sim N\left(\begin{pmatrix}\theta_0\\ \theta_0\end{pmatrix}, \begin{pmatrix}\sigma_0^2 & \sigma_0^2\\ \sigma_0^2 & \sigma_0^2+1\end{pmatrix}\right)$$

giving that $\theta|y$ is Gaussian with

$$\begin{aligned}E[\theta|y] =& \theta_0 + \frac{\sigma_0^2}{\sigma_0^2+1}(y-\theta_0)\\ =& \frac{\theta_0+\sigma_0^2 y}{\sigma_0^2+1}\\ \mathrm{Var}[\theta|y] =& \sigma_0^2 - \frac{\sigma_0^4}{\sigma_0^2+1} = \frac{\sigma_0^2}{\sigma_0^2+1}\end{aligned}$$

(c) We have

$$\frac{\theta - \frac{\theta_0+\sigma_0^2 y}{\sigma_0^2+1}}{\frac{\sigma_0}{\sqrt{\sigma_0^2+1}}} \sim N(0,1)$$

which gives

$$\Pr(\theta \leq 0) = \Pr\left(Z \leq -\frac{\frac{\theta_0+\sigma_0^2 y}{\sigma_0^2+1}}{\frac{\sigma_0}{\sqrt{\sigma_0^2+1}}}\right)$$

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to make this probability equal to 0.1, we need

$$-\frac{\frac{\theta_0+\sigma_0^2 y}{\sigma_0^2+1}}{\frac{\sigma_0}{\sqrt{\sigma_0^2+1}}} = -1.282$$

or

$$\frac{\theta_0+\sigma_0^2 y}{\sigma_0^2+1} = 1.282\frac{\sigma_0}{\sqrt{\sigma_0^2+1}}$$

or

$$y = \frac{1.282\sigma_0\sqrt{\sigma_0^2+1}-\theta_0}{\sigma_0^2} = 3.728$$

Problem 2

(a) We have

$$\begin{aligned}\ell(\theta) =& \sum_{i=1}^n[-\frac{1}{2}\theta y_i^2 + \log(\theta) + \log(y_i)]\\ \frac{\partial}{\partial\theta}\ell(\theta) =& \sum_{i=1}^n[-\frac{1}{2}y_i^2 + \frac{1}{\theta}] = -\frac{1}{2}\sum_{i=1}^n y_i^2 + \frac{n}{\theta}\\ \frac{\partial^2}{\partial\theta^2}\ell(\theta) =& -\frac{n}{\theta^2}\end{aligned}$$

so the function is unimodal and we obtain

$$\hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^n y_i^2} = 2w_n^{-1}$$

(b) We have

$$\begin{aligned}p(\theta|\boldsymbol{y}) \propto& \theta^{a-1}e^{-b\theta}\prod_{i=1}^n e^{-0.5\theta y_i^2}\theta y_i \propto \theta^{a+n-1}e^{[b+0.5\sum_{i=1}^n y_i^2]\theta}\\ \propto& \text{Gamma}(a_n, b_n)\end{aligned}$$

with

$$a_n = a+n \quad, b_n = b + 0.5\sum_{i=1}^n y_i^2$$

(c) Under square root loss, the Bayes rule is the posterior expectation, that is

$$\hat{\theta}_B = \frac{a+n}{b+0.5\sum_{i=1}^n y_i^2} = \frac{2a+2n}{2b+\sum_{i=1}^n y_i^2}$$

90% creidibility intervals can be found be taking the 0.05 and 0.95 quantiles in the Gamma distribution about.

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(d) We have

$$\begin{aligned}
p(y_{n+1}|\boldsymbol{y}) =& \int_\theta p(y_{n+1}|\theta)p(\theta|\boldsymbol{y})d\theta\\
=& \int_\theta e^{-0.5\theta y_{n+1}^2}\theta y_{n+1}\frac{b_n^{a_n}}{\Gamma(a_n)}\theta^{a_n-1}e^{-b_n\theta}d\theta\\
=& a_n y_{n+1}\frac{b_n^{a_n}}{[b_n+0.5y_{n+1}^2]^{a_n+1}}\times\\
& \int_\theta \frac{[b_n+0.5y_{n+1}^2]^{a_n+1}}{\Gamma(a_n+1)}\theta^{a_n+1-1}e^{-[b_n+0.5y_{n+1}^2]\theta}d\theta\\
=& a_n y_{n+1}\frac{b_n^{a_n}}{[b_n+0.5y_{n+1}^2]^{a_n+1}}
\end{aligned}$$

Problem 3

(a) We have

$$\begin{aligned}
\Pr(M_1|y) &\propto \Pr(M_1)p(y|M_1) = \pi_1 f_1(y)\\
\Pr(M_2|y) &\propto \Pr(M_2)p(y|M_2) = \pi_2 f_2(y)
\end{aligned}$$

Since these probabilities needs to sum to one, we obtain

$$\Pr(M_1|y) = \frac{\pi_1 f_1(y)}{\pi_1 f_1(y) + \pi_2 f_2(y)} = \frac{\pi_1}{\pi_1 + \pi_2 f_2(y)/f_1(y)}$$

(b) We have

$$\begin{aligned}
E[L(M,2)|y] &= 1\cdot\Pr(M=1|y) + 0\cdot\Pr(M=2|y) = 1-\Pr(M_2|y)\\
E[L(M,1)|y] &= 1\cdot\Pr(M=2|y) + 0\cdot\Pr(M=1|y) = 1-\Pr(M_1|y)
\end{aligned}$$

We then make decision 1 if $1-\Pr(M_1|y) < 1-\Pr(M_2|y)$ or $\Pr(M_1|y) > \Pr(M_2|y)$.

(c) We classify to normal if $f_1(y) > f_2(y)$, that is

$$\frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}y^2) > \frac{1}{\sqrt{2}}\exp(-\sqrt{2}|y|)$$

or

$$-\frac{1}{2}y^2 - \frac{1}{2}\log(\pi) > -\sqrt{2}|y|$$

or

$$y^2 < 2\sqrt{2}|y| - \log(\pi)$$

(Continued on page 4.)

(d) Now

$$\Pr(M_1|y_1,...,y_n) \propto \pi_1 \prod_{i=1}^n f_1(y_i)$$
$$\Pr(M_2|y_1,...,y_n) \propto \pi_2 \prod_{i=1}^n f_2(y_i)$$

(e) (Tricky!!) We have that

$$\frac{\Pr(M_1|\boldsymbol{y})}{\Pr(M_2|\boldsymbol{y})} = \exp(n\frac{1}{n}\sum_{i=1}^n \log f_1(y_i) - n\frac{1}{n}\sum_{i=1}^n \log f_2(y_i))$$
$$\approx \exp(n[E[\log f_1(y_i)] - E[\log f_2(y_i)]])$$

for large n

For $y_i \sim f_1$, we have

$$E[\log f_1(y_i)] - E[\log f_2(y_i)]] = \int_y f_1(y)\log\frac{f_1(y)}{f_2(y)}dy$$

which is the Kullback-Leibler distance between f_1 and f_2. This distance is always positive, making the fraction above increase exponentially and making $\Pr(M_1|y)$ convergence towards 1.

For $y_i \sim f_2$, we have

$$E[\log f_1(y_i)] - E[\log f_2(y_i)]] = \int_y f_2(y)\log\frac{f_1(y)}{f_2(y)}dy = -\int_y f_2(y)\log\frac{f_2(y)}{f_1(y)}dy$$

which is the minus Kullback-Leibler distance between f_2 and f_1. This distance is again always positive, making the fraction above decrease exponentially and making $\Pr(M_1|y)$ convergence towards 0.

Problem 4

(a) We have

$$p(y|\theta) = \prod_{i=1}^n p(y_i|\theta) = \prod_{i=1}^n \frac{1}{\theta}I(0 \le y_i \le \theta) = \theta^{-n}I(0 \le \min(y_i) \le \max(y_i) \le \theta)$$

Now since the maximum is larger than 1 and θ^{-n} will decrease with increasing values of $\theta > 1$, we find that the maximum likelihood estimate is $\max(y_i) = 4.294$.

(b) We now have

$$p(\theta|y) \propto p(\theta)p(y|\theta) \propto \theta^{-(n+1)}I(\theta \ge 4.294)$$

(Continued on page 5.)

Now

$$\int_{4.294}^{\infty} \theta^{-11} d\theta = [-\frac{1}{10}\theta^{-10}]_{4.294}^{\infty} = \frac{1}{10}4.294^{-10}$$

giving

$$p(\theta|\boldsymbol{y}) = 10 * 4.294^{10}\theta^{-11}I(\theta \geq 4.294)$$

For the absolute loss, we have that the Bayes estimate is the median. Now

$$\begin{aligned}\Pr(\theta \leq u|\boldsymbol{y}) = &\int_{4.294}^{u}[10 * 4.294^{10}\theta^{-11}]dy\\ =&10 * 4.294^{10}[-\frac{1}{10}\theta^{-10}]_{4.294}^{u}\\ =&10 * 4.294^{10}[\frac{1}{10}4.294^{-10} - \frac{1}{10}u^{-10}]\\ =&1 - 4.294^{10}u^{-10}\end{aligned}$$

and putting this to 0.5, we obtain $\hat{\theta} = 4.294 * 2^{1/10} = 4.602$