UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK4021 — Applied Bayesian Analysis and Numerical Methods

Day of examination: **Sketch of solution**

This problem set consists of 3 pages.

Appendices: Some useful formulas. List of distributions

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

(i)

$$p(\theta \mid y) = \text{Gamma}(\alpha + y_1, \beta + 1)$$

(ii)

$$p(\theta \mid y) = \text{Gamma}(\alpha + \sum_{i=1}^{n} y_i, \beta + n)$$

b

(i) $E[\theta \mid y] = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} = \frac{5+67+43+74+37+59}{\frac{1}{6}+5} \approx 55$

(ii) Squared error loss: $L(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2$.

(iii) $\text{Var}[\theta \mid y] = \frac{\alpha + \sum_{i=1}^{n} y_i}{(\beta + n)^2} = \frac{5+67+43+74+37+59}{\left(\frac{1}{6}+5\right)^2} \approx 10.68$

c

(i) We have

$$\begin{aligned}
p(\tilde{y} \mid y) &= \int_0^\infty p(\tilde{y}, \theta \mid y) d\theta = \int_0^\infty p(\tilde{y} \mid \theta) p(\theta \mid y) d\theta \\
&\propto \int_0^\infty \frac{1}{\tilde{y}!} \theta^{\tilde{y}} \exp\{-\theta\} \theta^{\alpha + \sum_{i=1}^n y_i - 1} \exp\{-(\beta + n)\theta\} d\theta \\
&= \frac{1}{\tilde{y}!} \int_0^\infty \underbrace{\theta^{\tilde{y} + \alpha + \sum_{i=1}^n y_i - 1} \exp\{-(\beta + n + 1)\theta\}}_{\text{Proportional to a Gamma}(\tilde{y}+\alpha+\sum_{i=1}^n y_i, \beta+n+1)\text{-distribution}} d\theta \\
&\propto \frac{1}{\tilde{y}!} \frac{\Gamma(\tilde{y} + \alpha + \sum_{i=1}^n y_i)}{(\beta + n + 1)^{\tilde{y} + \alpha + \sum_{i=1}^n y_i}} \propto \frac{1}{\tilde{y}!} \frac{(\tilde{y} + \alpha + \sum_{i=1}^n y_i - 1)!}{(\beta + n + 1)^{\tilde{y}}} \\
&\propto \binom{\tilde{y} + \alpha + \sum_{i=1}^n y_i - 1}{\alpha + \sum_{i=1}^n y_i - 1} \frac{1}{(\beta + n + 1)^{\tilde{y}}} \propto \text{Negative-binomial}(\alpha + \sum_{i=1}^n y_i, \beta + n)
\end{aligned}$$

(Continued on page 2.)

(ii)

$$E[\tilde{y} \mid y] = \frac{\alpha + \sum_{i=1}^{n} y_i}{\beta + n} \approx 55$$
$$\text{Var}[\tilde{y} \mid y] = \frac{\alpha + \sum_{i=1}^{n} y_i}{(\beta + n)^2}(\beta + n + 1) = \frac{5 + 67 + 43 + 74 + 37 + 59}{\left(\frac{1}{6} + 5\right)^2}\left(\frac{1}{6} + 5 + 1\right) = 65.84$$

Problem 2

Suppose $y_1, \ldots, y_n$ are independent and identically distributed observations with

$$y_i \sim N(\mu, \sigma^2),\ i = 1, \ldots, n$$
$$\mu \sim N(\mu_0, \tau^2)$$

where σ^2, μ_0, τ^2 are fixed, known quantities.

a

$$\begin{aligned}
p(\mu \mid y) &\propto p(\mu) \cdot \prod_{i=1}^{n} p(y_i \mid \mu, \sigma^2) \propto \exp\left\{-\frac{1}{2\tau^2}(\mu - \mu_0)^2\right\} \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu)^2\right\} \\
&= \exp\left\{-\frac{1}{2\tau^2\sigma^2}\left[\sigma^2(\mu^2 - 2\mu_0\mu + \mu_0^2) + \tau^2\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau^2\sigma^2}\left[(\sigma^2 + n\tau^2)\mu^2 - 2(\sigma^2\mu_0 + n\tau^2\bar{y})\mu)\right]\right\} \\
&\propto \exp\left\{-\frac{(\sigma^2 + n\tau^2)}{2\tau^2\sigma^2}\left(\mu - \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2 + n\tau^2}\right)^2\right\} \\
&\propto N\left(\mu \middle| \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2 + n\tau^2}, \frac{\tau^2\sigma^2}{\sigma^2 + n\tau^2}\right)
\end{aligned}$$

Hence

$$E[\mu \mid y] = \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2 + n\tau^2} = \frac{\frac{\mu_0}{\tau^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}$$
$$\text{Var}[\mu \mid y] = \frac{\tau^2\sigma^2}{\sigma^2 + n\tau^2} = \frac{1}{\frac{1}{\tau^2} + \frac{n}{\sigma^2}}.$$

b

(i)-(ii) We have

$$\begin{aligned}
p(\tilde{y} \mid y) &= \int_{-\infty}^{\infty} p(\tilde{y}, \mu \mid y)d\mu = \int_{-\infty}^{\infty} p(\tilde{y} \mid \theta)p(\theta \mid y)d\mu \\
&\propto \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}(\tilde{y} - \mu)^2\right\} \cdot \exp\left\{-\frac{(\sigma^2 + n\tau^2)}{2\tau^2\sigma^2}\left(\mu - \frac{\sigma^2\mu_0 + n\tau^2\bar{y}}{\sigma^2 + n\tau^2}\right)^2\right\}d\mu
\end{aligned} \tag{1}$$

(Continued on page 3.)

One way of solving it: The integrand could be manipulated to express a Normal distribution for μ given y and $\tilde{y}$, then integrate out μ to yield a Normal marginal posterior predictive distribution for $\tilde{y}$.

A quicker way: From (1) it is easy to see that $(\tilde{y}, \mu) \mid y$ has a bivariate Normal distribution, since the integrand is a the exponential of a quadratic function of $(\tilde{y}, \mu)$. Hence, integrating out μ yields a Normal marginal posterior predictive distribution for $\tilde{y}$. Then one can find the corresponding mean and variance in the following way

$$\begin{aligned}
\mathrm{E}\left(\tilde{y} \mid y\right) &= \mathrm{E}\left(\mathrm{E}\left(\tilde{y} \mid \mu, y\right) \mid y\right) = \mathrm{E}\left(\mu \mid y\right) = \frac{\sigma^2 \mu_0 + n\tau^2 \bar{y}}{\sigma^2 + n\tau^2} \\
\mathrm{Var}\left(\tilde{y} \mid y\right) &= \mathrm{E}\left(\mathrm{Var}\left(\tilde{y} \mid \mu, y\right) \mid y\right) + \mathrm{Var}\left(\mathrm{E}\left(\tilde{y} \mid \mu, y\right) \mid y\right) \\
&= \mathrm{E}\left(\sigma^2 \mid y\right) + \mathrm{Var}\left(\mu \mid y\right) = \sigma^2 + \frac{\tau^2 \sigma^2}{\sigma^2 + n\tau^2}
\end{aligned}$$

THE END