

Givens & Hoeting - solutions to exercises

Geir Storvik

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Solution to exercise (2.1)

The Cauchy($\theta, 1$) distribution is given by

$$f(x; \theta) = \frac{1}{\pi[1 + (x - \theta)^2]}$$

(a). We have

$$\begin{aligned}\ell(\theta) &= \sum_{i=1}^n [-\log(\pi) - \log(1 + (x_i - \theta)^2)] \\ \ell'(\theta) &= \sum_{i=1}^n \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2} \\ \ell''(\theta) &= - \sum_{i=1}^n \frac{1 - (x_i - \theta)^2}{[1 + (x_i - \theta)^2]^2}\end{aligned}$$

Solution to exercise (2.3)

We have

$$\log(S(t))' = \frac{S'(t)}{S(t)} = -\frac{f(t)}{S(t)} = -h(t) = -\lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta})$$

giving

$$\begin{aligned}\log(S(t)) &= -\Lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta}) \\ S(t) &= \exp\{-\Lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta})\} \\ f(t) &= -S'(t) = \lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta}) \exp\{-\Lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta})\}\end{aligned}$$

(a). If observation i is not censored, we have $w_i = 1$ and

$$\begin{aligned}
l_i &= \log(f(t_i)) = \log(\lambda(t_i)) + \mathbf{x}_i^T \boldsymbol{\beta} - \Lambda(t_i) \exp(\mathbf{x}_i^T \boldsymbol{\beta}) \\
&= \log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right) + \log(\Lambda(t_i)) + \mathbf{x}_i^T \boldsymbol{\beta} - \mu_i \\
&= \log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right) + \log(\Lambda(t_i) \exp(\mathbf{x}_i^T \boldsymbol{\beta})) - \mu_i \\
&= \log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right) + \log(\mu_i) - \mu_i \\
&= w_i \log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right) + w_i \log(\mu_i) - \mu_i
\end{aligned}$$

If observation i is censored, we have $w_i = 0$ and

$$\begin{aligned}
L_i &= \Pr(T \geq t_i) = S(t_i) \\
&= \exp\{-\lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta})\} \\
l_i &= -\lambda(t) \exp(\mathbf{x}_i^T \boldsymbol{\beta}) = -\mu_i \\
&= w_i \log(\mu_i) - \mu_i + w_i \log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right)
\end{aligned}$$

(b). We have in this case

$$\begin{aligned}
\frac{\partial}{\partial \alpha} \Lambda(t) &= \frac{\partial}{\partial \alpha} t^\alpha = \frac{\partial}{\partial \alpha} e^{\alpha \log(t)} = e^{\alpha \log(t)} \log(t) = \Lambda(t) \log(t) \\
\frac{\partial}{\partial \alpha} \mu_i &= \Lambda(t_i) \log(t_i) \exp(\mathbf{x}_i^T \boldsymbol{\beta}) = \mu_i \log(t_i) \\
\frac{\partial}{\partial \boldsymbol{\beta}} \mu_i &= \Lambda(t_i) \exp(\mathbf{x}_i^T \boldsymbol{\beta}) \mathbf{x}_i^T = \mu_i \mathbf{x}_i^T
\end{aligned}$$

giving

$$\begin{aligned}\frac{\partial}{\partial \alpha} l(\boldsymbol{\theta}) &= \sum_{i=1}^n [(w_i - \mu_i) \log(t_i) + w_i/\alpha] \\ \frac{\partial}{\partial \boldsymbol{\beta}} l(\boldsymbol{\theta}) &= \sum_{i=1}^n (w_i \frac{\mu_i \mathbf{x}_i}{\mu_i} - \mu_i \mathbf{x}_i) = \sum_{i=1}^n (w_i - \mu_i) \mathbf{x}_i \\ \frac{\partial^2}{\partial \alpha^2} l(\boldsymbol{\theta}) &= - \sum_{i=1}^n [\mu_i \log(t_i)^2 + w_i/\alpha^2] \\ \frac{\partial^2}{\partial \boldsymbol{\beta} \partial \alpha} l(\boldsymbol{\theta}) &= - \sum_{i=1}^n \mu_i \mathbf{x}_i \log(t_i) \\ \frac{\partial^2}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} l(\boldsymbol{\theta}) &= - \sum_{i=1}^n \mu_i \mathbf{x}_i \mathbf{x}_i^T\end{aligned}$$

Solution to exercise (2.4)

We need to find two points x_1, x_2 such that

$$\begin{aligned}s_1(x_1, x_2) &= f(x_2) - f(x_1) = 0 \\ s_2(x_1, x_2) &= F(x_2) - F(x_1) - 0.95 = 0\end{aligned}$$

with

$$\begin{aligned}f(x) &= xe^{-x} \\ F(x) &= \int_0^x f(u) du\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial}{\partial x_1} s_1(x_1, x_2) &= -f'(x_1) \\ \frac{\partial}{\partial x_2} s_1(x_1, x_2) &= f'(x_2) \\ \frac{\partial}{\partial x_1} s_2(x_1, x_2) &= -f(x_1) \\ \frac{\partial}{\partial x_2} s_2(x_1, x_2) &= f(x_2)\end{aligned}$$

were

$$f'(x) = e^{-x}[1 - x]$$

Solution to exercise (4.2)

(a). We introduce $\gamma = 1 - \alpha - \beta$ with the constraints $\alpha + \beta + \gamma = 1$. Complete likelihood:

$$l(\boldsymbol{\theta}) = n_{z,0} \log(\alpha) + \sum_{i=0}^{16} [n_{t,i}(\log(\beta) + i \log(\mu) - \mu) + n_{p,i}(\log(\gamma) + i \log(\lambda) - \lambda)]$$

Then (with $\mathbf{n} = (n_0, \dots, n_{16})$ and using s to denote iteration number in order not to confuse with t in model)

$$\begin{aligned} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) &= E[N_{z,0}|\mathbf{n}, \boldsymbol{\theta}^{(s)}] \log(\alpha) + \\ &\quad \sum_{i=0}^{16} [E[N_{t,i}|\mathbf{n}, \boldsymbol{\theta}^{(s)}](\log(\beta) + i \log(\mu) - \mu) + \\ &\quad \sum_{i=0}^{16} E[N_{p,i}|\mathbf{n}, \boldsymbol{\theta}^{(s)}](\log(\gamma) + i \log(\lambda) - \lambda)] \end{aligned}$$

Assume now an individual j has answered i and denote by G_j the group membership. Then

$$\begin{aligned} \Pr(G_j = t | x_j = i) &= \frac{\Pr(G_i = t) \Pr(x_j = i | G_j = t)}{\Pr(x_j = i)} \\ &= \frac{\beta \mu^i \exp(-\mu)}{\pi_i(\boldsymbol{\theta})} \end{aligned}$$

($i!$ is then deleted in both the nominator and the denominator) which leads to (using that the individuals are independent so that $N_{t,i}$ is binomial distributed with probability defined above)

$$E[N_{t,i}|\mathbf{n}, \boldsymbol{\theta}^{(s)}] = n_i \frac{\beta^{(s)} (\mu^{(s)})^i \exp(-\mu^{(s)})}{\pi_i(\boldsymbol{\theta}^{(s)})} = n_i t_i(\boldsymbol{\theta}^{(s)})$$

We similarly get

$$\begin{aligned} E[N_{z,0}|\mathbf{n}, \boldsymbol{\theta}^{(s)}] &= n_0 \frac{\alpha^{(s)}}{\pi_0(\boldsymbol{\theta}^{(s)})} = n_0 z_0(\boldsymbol{\theta}^{(s)}) \\ E[N_{p,i}|\mathbf{n}, \boldsymbol{\theta}^{(s)}] &= n_i \frac{\gamma^{(s)} (\lambda^{(s)})^i \exp(-\lambda^{(s)})}{\pi_i(\boldsymbol{\theta}^{(s)})} = n_i p_i(\boldsymbol{\theta}^{(s)}) \end{aligned}$$

Further, introducing the Lagrange term,

$$Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) + \phi(1 - \alpha - \beta - \gamma)$$

we get

$$\frac{\partial}{\partial \alpha} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = n_0 z_0(\boldsymbol{\theta}^{(s)}) \frac{1}{\alpha} - \phi$$

so

$$\alpha^{(s+1)} = \frac{1}{\phi} n_0 z_0(\boldsymbol{\theta}^{(s)})$$

$$\frac{\partial}{\partial \beta} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)}) \frac{1}{\beta} - \phi$$

so

$$\beta^{(s+1)} = \frac{1}{\phi} \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)})$$

$$\frac{\partial}{\partial \gamma} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=0}^{16} p_i(\boldsymbol{\theta}^{(s)}) \frac{1}{\gamma} - \phi$$

so

$$\gamma^{(s+1)} = \frac{1}{\phi} \sum_{i=0}^{16} n_i p_i(\boldsymbol{\theta}^{(s)})$$

By noting that

$$n_0 z_0(\boldsymbol{\theta}^{(s)}) + \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)}) + \sum_{i=0}^{16} n_i p_i(\boldsymbol{\theta}^{(s)}) = N$$

we get

$$\alpha^{(s+1)} = \frac{n_0 z_0(\boldsymbol{\theta}^{(s)})}{N}$$

$$\beta^{(s+1)} = \sum_{i=0}^{16} \frac{n_i t_i(\boldsymbol{\theta}^{(s)})}{N}$$

which corresponds to the formulas in the book.

Similarly,

$$\frac{\partial}{\partial \mu} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)}) \left(\frac{i}{\mu} - 1 \right)$$

giving

$$\mu^{(s+1)} = \frac{\sum_{i=0}^{16} in_i t_i(\boldsymbol{\theta}^{(s)})}{\sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)})}$$

and similarly

$$\lambda^{(s+1)} = \frac{\sum_{i=0}^{16} in_i p_i(\boldsymbol{\theta}^{(s)})}{\sum_{i=0}^{16} n_i p_i(\boldsymbol{\theta}^{(s)})}$$

Solution to exercise (4.3)

(a). Define \mathbf{y}_i to be the complete data and \mathbf{x}_i the observed data. Note in general that we have

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim N \left(\begin{pmatrix} \boldsymbol{\mu}_x \\ \boldsymbol{\mu}_y \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xy} \\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_y \end{pmatrix} \right)$$

then

$$\begin{aligned} E[\mathbf{y}|\mathbf{x}] &= \boldsymbol{\mu}_y - \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}(\mathbf{x} - \boldsymbol{\mu}_x) \\ V[\mathbf{y}|\mathbf{x}] &= \boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy} \end{aligned}$$

We have

$$\begin{aligned} L^{compl}(\boldsymbol{\theta}) &= \prod_{i=1}^n \frac{1}{(2\pi)^{3/2}|\boldsymbol{\Sigma}|^{1/2}} \exp(-0.5(\mathbf{y}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})) \\ l^{compl}(\boldsymbol{\theta}) &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \sum_{i=1}^n (\mathbf{y}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu}) \\ Q(\boldsymbol{\theta}, \boldsymbol{\theta}^t) &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \sum_{i=1}^n E[(\mathbf{y}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \sum_{i=1}^n E[(\mathbf{y}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \sum_{i=1}^n E[\text{tr}\{(\mathbf{y}_i - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})\}|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \sum_{i=1}^n E[\text{tr}\{\boldsymbol{\Sigma}^{-1}(\mathbf{y}_i - \boldsymbol{\mu})(\mathbf{y}_i - \boldsymbol{\mu})^T\}|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \sum_{i=1}^n \text{tr}\{\boldsymbol{\Sigma}^{-1} E[(\mathbf{y}_i - \boldsymbol{\mu})(\mathbf{y}_i - \boldsymbol{\mu})^T|\mathbf{x}_i, \boldsymbol{\theta}^t]\} \\ &= -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5 \boldsymbol{\Sigma}^{-1} \sum_{i=1}^n \text{tr}\{E[(\mathbf{y}_i - \boldsymbol{\mu})(\mathbf{y}_i - \boldsymbol{\mu})^T|\mathbf{x}_i, \boldsymbol{\theta}^t]\} \end{aligned}$$

Further,

$$\begin{aligned} &E[(\mathbf{y}_i - \boldsymbol{\mu})(\mathbf{y}_i - \boldsymbol{\mu})^T|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= E[(\mathbf{y}_i - E[\mathbf{y}_i|\mathbf{x}_i] + E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})(\mathbf{y}_i - E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t] + E[\mathbf{y}_i|\mathbf{x}_i] - \boldsymbol{\mu})^T|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= E[(\mathbf{y}_i - E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t])(\mathbf{y}_i - E[\mathbf{y}_i|\mathbf{x}_i])^T|\mathbf{x}_i, \boldsymbol{\theta}^t] + \\ &\quad E[(E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})(E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})^T|\mathbf{x}_i, \boldsymbol{\theta}^t] \\ &= V(\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t) + (E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})(E[\mathbf{y}_i|\mathbf{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})^T \\ &= \mathbf{V}_i^t + (\hat{\mathbf{y}}_i^t - \boldsymbol{\mu})(\hat{\mathbf{y}}_i^t - \boldsymbol{\mu})^T \end{aligned}$$

Inserting this, we get

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}^t) = -\frac{3n}{2} \log(2\pi) - \frac{n}{2} \log(|\boldsymbol{\Sigma}|) - 0.5\boldsymbol{\Sigma}^{-1} \sum_{i=1}^n \text{tr}\{\mathbf{V}_i^t + (\hat{\mathbf{y}}_i^t - \boldsymbol{\mu})(\hat{\mathbf{y}}_i^t - \boldsymbol{\mu})^T\}$$

Differentiating wrt $\boldsymbol{\mu}$, we get

$$\begin{aligned} \boldsymbol{\Sigma}^{-1} \sum_{i=1}^n (\hat{\mathbf{y}}_i^t - \boldsymbol{\mu}^{t+1}) &= \mathbf{0} \\ \sum_{i=1}^n (\hat{\mathbf{y}}_i^t - \boldsymbol{\mu}^{t+1}) &= \mathbf{0} \\ \boldsymbol{\mu}^{t+1} &= \frac{1}{n} \sum_{i=1}^n \hat{\mathbf{y}}_i^t \end{aligned}$$

Can similarly obtain

$$\boldsymbol{\Sigma}^{t+1} = \sum_{i=1}^n [\mathbf{V}_i^t + (\hat{\mathbf{y}}_i^t - \boldsymbol{\mu}^{t+1})(\hat{\mathbf{y}}_i^t - \boldsymbol{\mu}^{t+1})^T]$$

Solution to exercise (6.3)

(a). Use proposal $N(0, 1)$

(b).

Solution to exercise (7.6)

(a). We have

$$\begin{aligned}
p(\lambda_1, \lambda_2 | \dots) &\propto \frac{\alpha^3 \lambda_1^{3-1}}{\Gamma(3)} e^{-\alpha \lambda_1} \frac{\alpha^3 \lambda_2^{3-1}}{\Gamma(3)} e^{-\alpha \lambda_2} \times \\
&\quad \prod_{j=1}^{\theta} \frac{\lambda_1^{x_j - \lambda_1}}{e} x_j! \prod_{j=\theta+1}^{112} \frac{\lambda_2^{x_j} e^{-\lambda_2}}{x_j!} \\
&\propto \lambda_1^{2 + \sum_{j=1}^{\theta} x_j} e^{-(\alpha + \theta) \lambda_1} \lambda_2^{2 + \sum_{j=\theta+1}^{112} x_j} e^{-(\alpha + \theta) \lambda_2} \\
&\propto \text{Gamma}(\lambda_1; 3 + \sum_{j=1}^{\theta} x_j, \alpha + \theta) \text{Gamma}(\lambda_2; 3 + \sum_{j=\theta+1}^{112} x_j, \alpha + 112 - \theta) \\
p(\alpha | \dots) &\propto \frac{10^{10} \alpha^{10-1}}{\Gamma(10)} e^{-10\alpha} \frac{\alpha^3 \lambda_1^{3-1}}{\Gamma(3)} e^{-\alpha \lambda_1} \frac{\alpha^3 \lambda_2^{3-1}}{\Gamma(3)} e^{-\alpha \lambda_2} \\
&\propto \alpha^{15} e^{-(10 + \lambda_1 + \lambda_2) \alpha} \\
&\propto \text{Gamma}(\alpha; 16, 10 + \lambda_1 + \lambda_2) \\
p(\theta) &\propto \frac{1}{111} \prod_{j=1}^{\theta} \frac{\lambda_1^{x_j - \lambda_1}}{e} x_j! \prod_{j=\theta+1}^{112} \frac{\lambda_2^{x_j} e^{-\lambda_2}}{x_j!} \\
&\propto \lambda_1^{\sum_{j=1}^{\theta} x_j} e^{-\theta \lambda_1} \lambda_2^{\sum_{j=\theta+1}^{112} x_j} e^{-\theta \lambda_2}
\end{aligned}$$

(b).

Solution to exercise (7.7)

(a). We have

$$\begin{aligned}
& p(\mu|\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}) \\
& \propto p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}) \\
& = p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\mathbf{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \propto p(\mu)p(\mathbf{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \propto \prod_{i=1}^I \prod_{j=1}^{J_i} \exp\left[-\frac{1}{2\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i - \beta_{j(i)})^2\right] \\
& \propto \prod_{i=1}^I \prod_{j=1}^{J_i} \exp\left[-\frac{1}{2\sigma_\varepsilon^2}[(y_{ij} - \alpha_i - \beta_{j(i)})^2 - 2(y_{ij} - \alpha_i - \beta_{j(i)})\mu + \mu^2]\right] \\
& \propto \exp\left[-\frac{1}{2\sigma_\varepsilon^2}\left[n\mu^2 - 2\sum_{i=1}^I \sum_{j=1}^{J_i} (y_{ij} - \alpha_i - \beta_{j(i)})\mu\right]\right] \\
& \propto \exp\left[-\frac{n}{2\sigma_\varepsilon^2}\left[\mu - \frac{1}{n}\sum_{i=1}^I \sum_{j=1}^{J_i} (y_{ij} - \alpha_i - \beta_{j(i)})\right]^2\right] \\
& \propto \exp\left[-\frac{n}{2\sigma_\varepsilon^2}\left[\mu - \left(y_{..} - \frac{1}{n}\sum_{i=1}^I J_i\alpha_i - \frac{1}{n}\sum_{i=1}^I \sum_{j=1}^{J_i} \beta_{j(i)}\right)\right]^2\right]
\end{aligned}$$

showing the first result. Similarly,

$$\begin{aligned}
& p(\alpha_i|\mu, \boldsymbol{\alpha}_{-i}, \boldsymbol{\beta}, \mathbf{y}) \\
& \propto p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}) \\
& = p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\mathbf{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \propto p(\alpha_i)p(\mathbf{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \propto \exp\left(-\frac{1}{2\sigma_\alpha^2}\alpha_i^2\right) \prod_{j=1}^{J_i} \exp\left[-\frac{1}{2\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i - \beta_{j(i)})^2\right] \\
& \propto \exp\left(-\frac{1}{2\sigma_\alpha^2}\alpha_i^2\right) \prod_{j=1}^{J_i} \exp\left[-\frac{1}{2\sigma_\varepsilon^2}[(y_{ij} - \mu - \beta_{j(i)})^2 - 2(y_{ij} - \mu - \beta_{j(i)})\alpha_i + \alpha_i^2]\right] \\
& \propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\alpha^2}\alpha_i^2 + \frac{1}{\sigma_\varepsilon^2}J_i\alpha_i^2 - 2\frac{1}{\sigma_\varepsilon^2}\sum_{j=1}^{J_i} (y_{ij} - \mu - \beta_{j(i)})\alpha_i\right]\right] \\
& \propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\varepsilon^2}\right]\left[\alpha_i - \frac{1}{\sigma_\varepsilon^2} \frac{\sum_{j=1}^{J_i} (y_{ij} - \mu - \beta_{j(i)})}{\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\varepsilon^2}}\right]^2\right] \\
& \propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\varepsilon^2}\right]\left[\alpha_i - J_i \frac{1}{\sigma_\varepsilon^2} \frac{y_{i..} - \mu - \frac{1}{J_i} \sum_{j=1}^{J_i} \beta_{j(i)}}{\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\varepsilon^2}}\right]^2\right]
\end{aligned}$$

giving the second result. Finally,

$$\begin{aligned}
& p(\beta_{j(i)}|\mu, \boldsymbol{\alpha}_{-i}, \boldsymbol{\beta}, \mathbf{y}) \\
& \propto p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{y}) \\
& = p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\mathbf{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \propto p(\beta_{j(i)})p(\mathbf{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \propto \exp\left(-\frac{1}{2\sigma_\beta^2}\beta_{j(i)}^2\right) \exp\left[-\frac{1}{2\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i - \beta_{j(i)})^2\right] \\
& \propto \exp\left(-\frac{1}{2\sigma_\beta^2}\beta_{j(i)}^2\right) \exp\left[-\frac{1}{2\sigma_\varepsilon^2}[(y_{ij} - \mu - \alpha_i)^2 - 2(y_{ij} - \mu - \alpha_i)\beta_{j(i)} + \beta_{j(i)}^2]]\right] \\
& \propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\beta^2}\beta_{j(i)}^2 + \frac{1}{\sigma_\varepsilon^2}\beta_{j(i)}^2 - 2\frac{1}{\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i)\beta_{j(i)}\right]\right] \\
& \propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}\right]\left[\beta_{j(i)} - \frac{1}{\sigma_\varepsilon^2}\frac{(y_{ij} - \mu - \alpha_i)}{\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}}\right]^2\right] \\
& \propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}\right]\left[\beta_{j(i)} - \frac{1}{\sigma_\varepsilon^2}\frac{y_{ij} - \mu - \alpha_i}{\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}}\right]^2\right]
\end{aligned}$$

(b). The now model is now

$$\begin{aligned}
Y_{ij} &= \eta_{ij} + \varepsilon_{ij} \\
\eta_{ij} &\sim N(\gamma_i, \sigma_\beta^2) \\
\gamma_i &\sim N(\mu, \sigma_\alpha^2) \\
f(\mu) &\propto 1
\end{aligned}$$

We then have

$$\begin{aligned}
p(\mu|\boldsymbol{\gamma}, \boldsymbol{\eta}, \mathbf{y}) &\propto p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \mathbf{y}) = p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\boldsymbol{\gamma})p(\mathbf{y}|\boldsymbol{\eta}) \\
&\propto p(\mu)p(\boldsymbol{\gamma}|\mu) \\
&\propto \prod_{i=1}^I \exp\left[-\frac{1}{2\sigma_\alpha^2}(\gamma_i - \mu)^2\right] = \exp\left[-\frac{1}{2\sigma_\alpha^2}\left[I\mu^2 - 2\sum_{i=1}^I \gamma_i^2\mu\right]\right] \\
&\propto \exp\left[-\frac{I}{2\sigma_\alpha^2}\left[\mu - \frac{1}{I}\sum_{i=1}^I J_i\gamma_i\right]^2\right]
\end{aligned}$$

Further,

$$\begin{aligned}
p(\gamma_i|\mu, \boldsymbol{\gamma}_{-i}, \boldsymbol{\beta}, \mathbf{y}) &\propto p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \mathbf{y}) \\
&= p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\boldsymbol{\gamma})p(\mathbf{y}|\boldsymbol{\eta}) \propto (\gamma_i|\mu)p(\boldsymbol{\eta}_i|\gamma_i) \\
&\propto \exp\left(-\frac{1}{2\sigma_\alpha^2}(\gamma_i - \mu)^2\right) \prod_{j=1}^{J_i} \exp\left[-\frac{1}{2\sigma_\beta^2}(\eta_{ij} - \gamma_i)^2\right] \\
&\propto \exp\left[-\frac{1}{2}\left[\left(\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\beta^2}\right)\gamma_i^2 - 2\left(\frac{1}{\sigma_\alpha^2}\mu + \frac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij}\right)\gamma_i\right]\right] \\
&\propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\beta^2}\right]\left[\gamma_i - \frac{\frac{1}{\sigma_\alpha^2}\mu + \frac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij}}{\frac{1}{\sigma_\alpha^2} + \frac{1}{\sigma_\beta^2}}\right]^2\right]
\end{aligned}$$

and finally

$$\begin{aligned}
p(\eta_{ij}|\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}_{-ij}, \mathbf{y}) &\propto p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \mathbf{y}) \\
&= p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\boldsymbol{\gamma})p(\mathbf{y}|\boldsymbol{\eta}) \propto p(\eta_{ij})p(y_{ij}|\eta_{ij}) \\
&\propto \exp\left(-\frac{1}{2\sigma_\beta^2}(\eta_{ij} - \gamma_i)^2\right) \exp\left[-\frac{1}{2\sigma_\varepsilon^2}(y_{ij} - \eta_{ij})^2\right] \\
&\propto \exp\left[-\frac{1}{2}\left[\left(\frac{1}{\sigma_\beta^2} + \frac{1}{2\sigma_\varepsilon^2}\right)\eta_{ij}^2 - 2\left(\frac{1}{\sigma_\beta^2}\gamma_i + \frac{1}{\sigma_\varepsilon^2}y_{ij}\right)\eta_{ij}\right]\right] \\
&\propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}\right]\left[\eta_{ij} - \frac{\frac{1}{\sigma_\beta^2}\gamma_i + \frac{1}{\sigma_\varepsilon^2}y_{ij}}{\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}}\right]^2\right] \\
&\propto \exp\left[-\frac{1}{2}\left[\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}\right]\left[\beta_{j(i)} - \frac{1}{\sigma_\varepsilon^2}\frac{y_{ij} - \mu - \alpha_i}{\frac{1}{\sigma_\alpha^2} + \frac{1}{\sigma_\beta^2}}\right]^2\right]
\end{aligned}$$