Givens & Hoeting - solutions to exercises

Geir Storvik

Spring 2020

Solution to exercise (2.1)
The Cauchy$(\theta, 1)$ distribution is given by

$$f(x;\theta) = \frac{1}{\pi[1+(x-\theta)^2]}$$

(a). We have

$$\begin{aligned}
\ell(\theta) =& \sum_{i=1}^n [-\log(\pi) - \log(1+(x_i-\theta)^2)]\\
\ell'(\theta) =& \sum_{i=1}^n \frac{2(x_i-\theta)}{1+(x_i-\theta)^2}\\
\ell''(\theta) =& -\sum_{i=1}^n \frac{1-(x_i-\theta)^2}{[1+(x_i-\theta)^2]^2}
\end{aligned}$$

Solution to exercise (2.3)
We have

$$\log(S(t))' = \frac{S'(t)}{S(t)} = -\frac{f(t)}{S(t)} = -h(t) = -\lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$$

giving

$$\begin{aligned}
\log(S(t)) =& -\Lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})\\
S(t) =& \exp\{-\Lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})\}\\
f(t) =& -S'(t) = \lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})\exp\{-\Lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})\}
\end{aligned}$$

(a). If observation i is not censored, we have $w_i = 1$ and

$$\begin{aligned}
l_i =& \log(f(t_i)) = \log(\lambda(t_i)) + \boldsymbol{x}_i^T\boldsymbol{\beta} - \Lambda(t_i)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta}) \\
=& \log(\frac{\lambda(t_i)}{\Lambda(t_i)}) + \log(\Lambda(t_i)) + \boldsymbol{x}_i^T\boldsymbol{\beta} - \mu_i \\
=& \log(\frac{\lambda(t_i)}{\Lambda(t_i)}) + \log(\Lambda(t_i)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})) - \mu_i \\
=& \log(\frac{\lambda(t_i)}{\Lambda(t_i)}) + \log(\mu_i) - \mu_i \\
=& w_i \log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right) + w_i\log(\mu_i) - \mu_i
\end{aligned}$$

If observation i is censored, we have $w_i = 0$ and

$$\begin{aligned}
L_i =& \Pr(T \geq t_i) = S(t_i) \\
=& \exp\{-\Lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})\} \\
l_i =& -\Lambda(t)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta}) = -\mu_i \\
=& w_i\log(\mu_i) - \mu_i + w_i\log\left(\frac{\lambda(t_i)}{\Lambda(t_i)}\right)
\end{aligned}$$

(b). We have in this case

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\Lambda(t) =& \frac{\partial}{\partial\alpha}t^\alpha = \frac{\partial}{\partial\alpha}e^{\alpha\log(t)} = e^{\alpha\log(t)}\log(t) = \Lambda(t)\log(t) \\
\frac{\partial}{\partial\alpha}\mu_i =& \Lambda(t_i)\log(t_i)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta}) = \mu_i\log(t_i) \\
\frac{\partial}{\partial\boldsymbol{\beta}}\mu_i =& \Lambda(t_i)\exp(\boldsymbol{x}_i^T\boldsymbol{\beta})\boldsymbol{x}_i^T = \mu_i\boldsymbol{x}_i^T
\end{aligned}$$

giving

$$\frac{\partial}{\partial \alpha} l(\boldsymbol{\theta}) = \sum_{i=1}^{n} [(w_i - \mu_i)\log(t_i) + w_i/\alpha]$$
$$\frac{\partial}{\partial \boldsymbol{\beta}} l(\boldsymbol{\theta}) = \sum_{i=1}^{n} (w_i \frac{\mu_i \boldsymbol{x}_i}{\mu_i} - \mu_i \boldsymbol{x}_i) = \sum_{i=1}^{n} (w_i - \mu_i)\boldsymbol{x}_i$$
$$\frac{\partial^2}{\partial \alpha^2} l(\boldsymbol{\theta}) = -\sum_{i=1}^{n} [\mu_i \log(t_i)^2 + w_i/\alpha^2]$$
$$\frac{\partial^2}{\partial \boldsymbol{\beta} \partial \boldsymbol{\alpha}} l(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i \log(t_i)$$
$$\frac{\partial^2}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}^T} l(\boldsymbol{\theta}) = -\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i \boldsymbol{x}_i^T$$

Solution to exercise (2.4)
We need to find two points x_1, x_2 such that

$$s_1(x_1, x_2) = f(x_2) - f(x_1) = 0$$
$$s_2(x_1, x_2) = F(x_2) - F(x_1) - 0.95 = 0$$

with

$$f(x) = xe^{-x}$$
$$F(x) = \int_0^x f(u)du$$

Now

$$\frac{\partial}{\partial x_1} s_1(x_1, x_2) = -f'(x_1)$$
$$\frac{\partial}{\partial x_2} s_1(x_1, x_2) = f'(x_2)$$
$$\frac{\partial}{\partial x_1} s_2(x_1, x_2) = -f(x_1)$$
$$\frac{\partial}{\partial x_2} s_2(x_1, x_2) = f(x_2)$$

were

$$f'(x) = e^{-x}[1 - x]$$

Solution to exercise (4.2)

(a). We introduce $\gamma = 1-\alpha-\beta$ with the constraints $\alpha+\beta+\gamma = 1$. Complete likelihood:

$$l(\boldsymbol{\theta}) = n_{z,0}\log(\alpha) + \sum_{i=0}^{16}[n_{t,i}(\log(\beta) + i\log(\mu) - \mu) + n_{p,i}(\log(\gamma) + i\log(\lambda) - \lambda)]$$

Then (with $\boldsymbol{n} = (n_0, ..., n_{16})$ and using s to denote iteration number in order not to confuse with t in model)

$$\begin{aligned}Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) =&E[N_{z,0}|\boldsymbol{n},\boldsymbol{\theta}^{(s)}]\log(\alpha)+\\ &\sum_{i=0}^{16}[E[N_{t,i}|\boldsymbol{n},\boldsymbol{\theta}^{(s)}](\log(\beta) + i\log(\mu) - \mu)+\\ &\sum_{i=0}^{16}E[N_{p,i}|\boldsymbol{n},\boldsymbol{\theta}^{(s)}](\log(\gamma) + i\log(\lambda) - \lambda)]\end{aligned}$$

Assume now an individual j has answered i and denote by G_j the group membership. Then

$$\begin{aligned}&\Pr(G_j = t|x_j = i)\\ =&\frac{\Pr(G_i = t)\Pr(x_j = i|G_j = t)}{\Pr(x_j = i)}\\ =&\frac{\beta\mu^i\exp(-\mu)}{\pi_i(\boldsymbol{\theta})}\end{aligned}$$

($i!$ is then deleted in both the nominator and the denominator) which leads to (using that the individuals are independent so that $N_{t,i}$ is binomial distributed with probability defined above)

$$E[N_{t,i}|\boldsymbol{n},\boldsymbol{\theta}^{(s)}] =n_i\frac{\beta^{(s)}(\mu^{(s)})^i\exp(-\mu^{(s)})}{\pi_i(\boldsymbol{\theta}^{(s)})} = n_it_i(\boldsymbol{\theta}^{(s)})$$

We similarly get

$$\begin{aligned}E[N_{z,0}|\boldsymbol{n},\boldsymbol{\theta}^{(s)}] =&n_0\frac{\alpha^{(s)}}{\pi_0(\boldsymbol{\theta}^{(s)})} = n_0z_0(\boldsymbol{\theta}^{(s)})\\ E[N_{p,i}|\boldsymbol{n},\boldsymbol{\theta}^{(s)}] =&n_i\frac{\gamma^{(s)}(\lambda^{(s)})^i\exp(-\lambda^{(s)})}{\pi_i(\boldsymbol{\theta}^{(s)})} = n_ip_i(\boldsymbol{\theta}^{(s)})\end{aligned}$$

Further, introducing the Lagrange term,

$$Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) + \phi(1-\alpha-\beta-\gamma)$$

we get

$$\frac{\partial}{\partial \alpha} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = n_0 z_0(\boldsymbol{\theta}^{(s)}) \frac{1}{\alpha} - \phi$$

so

$$\alpha^{(s+1)} = \frac{1}{\phi} n_0 z_0(\boldsymbol{\theta}^{(s)})$$

$$\frac{\partial}{\partial \beta} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)}) \frac{1}{\beta} - \phi$$

so

$$\beta^{(s+1)} = \frac{1}{\phi} \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)})$$

$$\frac{\partial}{\partial \gamma} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=0}^{16} p_i(\boldsymbol{\theta}^{(s)}) \frac{1}{\gamma} - \phi$$

so

$$\gamma^{(s+1)} = \frac{1}{\phi} \sum_{i=0}^{16} n_i p_i(\boldsymbol{\theta}^{(s)})$$

By noting that

$$n_0 z_0(\boldsymbol{\theta}^{(s)}) + \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)}) + \sum_{i=0}^{16} n_i p_i(\boldsymbol{\theta}^{(s)}) = N$$

we get

$$\alpha^{(s+1)} = \frac{n_0 z_0(\boldsymbol{\theta}^{(s)})}{N}$$

$$\beta^{(s+1)} = \sum_{i=0}^{16} \frac{n_i t_i(\boldsymbol{\theta}^{(s)})}{N}$$

which corresponds to the formulas in the book.

Similarly,

$$\frac{\partial}{\partial \mu} Q_{lagr}(\boldsymbol{\theta}|\boldsymbol{\theta}^{(s)}) = \sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)}) (\frac{i}{\mu} - 1)$$

giving

$$\mu^{(s+1)} = \frac{\sum_{i=0}^{16} i n_i t_i(\boldsymbol{\theta}^{(s)})}{\sum_{i=0}^{16} n_i t_i(\boldsymbol{\theta}^{(s)})}$$

and similarly

$$\lambda^{(s+1)} = \frac{\sum_{i=0}^{16} i n_i p_i(\boldsymbol{\theta}^{(s)})}{\sum_{i=0}^{16} n_i p_i(\boldsymbol{\theta}^{(s)})}$$

Solution to exercise (4.3)

(a). Define $\boldsymbol{y}_i$ to be the complete data and $\boldsymbol{x}_i$ the observed data. Note in general that we have

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \sim N\left(\begin{pmatrix}\boldsymbol{\mu}_x\\ \boldsymbol{\mu}_y\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_{xx} & \boldsymbol{\Sigma}_{xy}\\ \boldsymbol{\Sigma}_{yx} & \boldsymbol{\Sigma}_{y}\end{pmatrix}\right)$$

then

$$\begin{aligned}E[\boldsymbol{y}|\boldsymbol{x}] =& \boldsymbol{\mu}_y + \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_x)\\ V[\boldsymbol{y}|\boldsymbol{x}] =& \boldsymbol{\Sigma}_{yy} - \boldsymbol{\Sigma}_{yx}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{xy}\end{aligned}$$

The trace operator,tr$\{\cdot\}$ is the sum of the diagonal elements of a matrix. The trace is a linear operator, meaning that it can swap place with sums and expectations. In addition, use relations (assuming that the dimensions match up):

$$\begin{aligned}\mathrm{tr}\{\boldsymbol{ABC}\} =& \mathrm{tr}\{\boldsymbol{BCA}\} = \mathrm{tr}\{\boldsymbol{CAB}\}\\ \frac{\partial \log(|\boldsymbol{\Sigma}|)}{\partial \boldsymbol{\Sigma}} =& \boldsymbol{\Sigma}^{-1}\\ \frac{\partial \mathrm{tr}\left\{\boldsymbol{A}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right\}}{\partial \boldsymbol{\Sigma}} =& -\left(\boldsymbol{\Sigma}^{-1}\boldsymbol{BA}\boldsymbol{\Sigma}^{-1}\right)^T\end{aligned}$$

We have

$$\begin{aligned}L^{compl}(\boldsymbol{\theta}) =& \prod_{i=1}^{n}\frac{1}{(2\pi)^{3/2}|\boldsymbol{\Sigma}|^{1/2}} - \exp\left(-0.5(\boldsymbol{y}_i-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i-\boldsymbol{\mu})\right)\\ l^{compl}(\boldsymbol{\theta}) =& -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\sum_{i=1}^{n}(\boldsymbol{y}_i-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i-\boldsymbol{\mu})\\ Q(\boldsymbol{\theta},\boldsymbol{\theta}^t) =& -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\sum_{i=1}^{n}E[(\boldsymbol{y}_i-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i-\boldsymbol{\mu})|\boldsymbol{x}_i,\boldsymbol{\theta}^t]\\ =& -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\sum_{i=1}^{n}E[\mathrm{tr}\{(\boldsymbol{y}_i-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i-\boldsymbol{\mu})\}|\boldsymbol{x}_i,\boldsymbol{\theta}^t]\\ =& -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\sum_{i=1}^{n}E[\mathrm{tr}\{\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i-\boldsymbol{\mu})(\boldsymbol{y}_i-\boldsymbol{\mu})^T\}|\boldsymbol{x}_i,\boldsymbol{\theta}^t]\\ =& -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\sum_{i=1}^{n}\mathrm{tr}\{\boldsymbol{\Sigma}^{-1}E[(\boldsymbol{y}_i-\boldsymbol{\mu})(\boldsymbol{y}_i-\boldsymbol{\mu})^T|\boldsymbol{x}_i,\boldsymbol{\theta}^t]\}\\ =& -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\mathrm{tr}\{\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{n}E[(\boldsymbol{y}_i-\boldsymbol{\mu})(\boldsymbol{y}_i-\boldsymbol{\mu})^T|\boldsymbol{x}_i,\boldsymbol{\theta}^t]\}\end{aligned}$$

Further,

$$
\begin{aligned}
E[(\boldsymbol{y}_i - \boldsymbol{\mu})(\boldsymbol{y}_i - \boldsymbol{\mu})^T|\boldsymbol{x}_i, \boldsymbol{\theta}^t]& \\
=&E[(\boldsymbol{y}_i - E[\boldsymbol{y}_i|\boldsymbol{x}_i] + E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})(\boldsymbol{y}_i - E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t] + E[\boldsymbol{y}_i|\boldsymbol{x}_i] - \boldsymbol{\mu})^T|\boldsymbol{x}_i, \boldsymbol{\theta}^t] \\
=&E[(\boldsymbol{y}_i - E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t])(\boldsymbol{y}_i - E[\boldsymbol{y}_i|\boldsymbol{x}_i])^T|\boldsymbol{x}_i, \boldsymbol{\theta}^t]+ \\
&E[(E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})(E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})^T|\boldsymbol{x}_i, \boldsymbol{\theta}^t] \\
=&V(\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t) + (E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})(E[\boldsymbol{y}_i|\boldsymbol{x}_i, \boldsymbol{\theta}^t] - \boldsymbol{\mu})^T \\
=&\boldsymbol{V}_i^t + (\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu})(\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu})^T
\end{aligned}
$$

Inserting this, we get

$$
Q(\boldsymbol{\theta}, \boldsymbol{\theta}^t) = -\frac{3n}{2}\log(2\pi) - \frac{n}{2}\log(|\boldsymbol{\Sigma}|) - 0.5\mathrm{tr}\{\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{n}\boldsymbol{V}_i^t + (\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu})(\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu})^T\}
$$

Differentiating wrt $\boldsymbol{\mu}$, we get

$$
\begin{aligned}
\boldsymbol{\Sigma}^{-1}\sum_{i=1}^{n}(\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu}^{t+1}) =&\boldsymbol{0} \\
\sum_{i=1}^{n}(\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu}^{t+1}) =&\boldsymbol{0} \\
\boldsymbol{\mu}^{t+1} =&\frac{1}{n}\sum_{i=1}^{n}\hat{\boldsymbol{y}}_i^t
\end{aligned}
$$

Can similarly obtain

$$
\boldsymbol{\Sigma}^{t+1} = \sum_{i=1}^{n}[\boldsymbol{V}_i^t + (\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu}^{t+1})(\hat{\boldsymbol{y}}_i^t - \boldsymbol{\mu}^{t+1})^T]
$$

Solution to exercise (6.3)

(a). Use proposal $N(0,1)$

Solution to exercise (7.6)
(a). We have

$$
\begin{aligned}
p(\lambda_1, \lambda_2 | \cdots) \propto & \frac{\alpha^3 \lambda_1^{3-1}}{\Gamma(3)} e^{-\alpha\lambda_1} \frac{\alpha^3 \lambda_2^{3-1}}{\Gamma(3)} e^{-\alpha\lambda_2} \times \\
& \prod_{j=1}^{\theta} \frac{\lambda_1^{x_j} e^{-\lambda_1}}{x_j!} \prod_{j=\theta+1}^{112} \frac{\lambda_2^{x_j} e^{-\lambda_2}}{x_j!} \\
\propto & \lambda_1^{2+\sum_{j=1}^{\theta} x_j} e^{-(\alpha+\theta)\lambda_1} \lambda_2^{2+\sum_{j=\theta+1}^{112} x_j} e^{-(\alpha+\theta)\lambda_2} \\
\propto & \text{Gamma}(\lambda_1; 3 + \sum_{j=1}^{\theta} x_j, \alpha + \theta) \text{Gamma}(\lambda_2; 3 + \sum_{j=\theta+1}^{112} x_j, \alpha + 112 - \theta)
\end{aligned}
$$

$$
\begin{aligned}
p(\alpha | \cdots) \propto & \frac{10^{10} \alpha^{10-1}}{\Gamma(10)} e^{-10\alpha} \frac{\alpha^3 \lambda_1^{3-1}}{\Gamma(3)} e^{-\alpha\lambda_1} \frac{\alpha^3 \lambda_2^{3-1}}{\Gamma(3)} e^{-\alpha\lambda_2} \\
\propto & \alpha^{15} e^{-(10+\lambda_1+\lambda_2)\alpha} \\
\propto & \text{Gamma}(\alpha; 16, 10 + \lambda_1 + \lambda_2)
\end{aligned}
$$

$$
\begin{aligned}
p(\theta | ...) \propto & \frac{1}{111} \prod_{j=1}^{\theta} \frac{\lambda_1^{x_j} e^{-\lambda_1}}{x_j!} \prod_{j=\theta+1}^{112} \frac{\lambda_2^{x_j} e^{-\lambda_2}}{x_j!} \\
\propto & \lambda_1^{\sum_{j=1}^{\theta} x_j} e^{-\theta\lambda_1} \lambda_2^{\sum_{j=\theta+1}^{112} x_j} e^{-(111-\theta)\lambda_2}
\end{aligned}
$$

Solution to exercise (7.7)
(a). We have

$$
\begin{aligned}
&p(\mu|\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y})\\
&\quad\propto p(\mu,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y})\\
&\quad= p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\boldsymbol{y}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\quad\propto p(\mu)p(\boldsymbol{y}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\quad\propto \prod_{i=1}^{I}\prod_{j=1}^{J_i}\exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij}-\mu-\alpha_i-\beta_{j(i)})^2]\\
&\quad\propto \prod_{i=1}^{I}\prod_{j=1}^{J_i}\exp[-\tfrac{1}{2\sigma_\varepsilon^2}[(y_{ij}-\alpha_i-\beta_{j(i)})^2-2(y_{ij}-\alpha_i-\beta_{j(i)})\mu+\mu^2]]\\
&\quad\propto \exp[-\tfrac{1}{2\sigma_\varepsilon^2}[n\mu^2-2\sum_{i=1}^{I}\sum_{j=1}^{J_i}(y_{ij}-\alpha_i-\beta_{j(i)})\mu]\\
&\quad\propto \exp[-\tfrac{n}{2\sigma_\varepsilon^2}[\mu-\tfrac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}(y_{ij}-\alpha_i-\beta_{j(i)})]^2]\\
&\quad\propto \exp[-\tfrac{n}{2\sigma_\varepsilon^2}[\mu-(y_{\cdot\cdot}-\tfrac{1}{n}\sum_{i=1}^{I}J_i\alpha_i-\tfrac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}\beta_{j(i)})]^2]
\end{aligned}
$$

showing the first result. Similarly,

$$
\begin{aligned}
&p(\alpha_i|\mu,\boldsymbol{\alpha}_{-i},\boldsymbol{\beta},\boldsymbol{y})\\
&\quad\propto p(\mu,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y})\\
&\quad= p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\boldsymbol{y}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\quad\propto p(\alpha_i)p(\boldsymbol{y}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\quad\propto \exp(-\tfrac{1}{2\sigma_\alpha^2}\alpha_i^2)\prod_{j=1}^{J_i}\exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij}-\mu-\alpha_i-\beta_{j(i)})^2]\\
&\quad\propto \exp(-\tfrac{1}{2\sigma_\alpha^2}\alpha_i^2)\prod_{j=1}^{J_i}\exp[-\tfrac{1}{2\sigma_\varepsilon^2}[(y_{ij}-\mu-\beta_{j(i)})^2-2(y_{ij}-\mu-\beta_{j(i)})\alpha_i+\alpha_i^2]]\\
&\quad\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2}\alpha_i^2+\tfrac{1}{\sigma_\varepsilon^2}J_i\alpha_i^2-2\tfrac{1}{\sigma_\varepsilon^2}\sum_{j=1}^{J_i}(y_{ij}-\mu-\beta_{j(i)})\alpha_i]\\
&\quad\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2}+\tfrac{J_i}{\sigma_\varepsilon^2}][\alpha_i-\tfrac{1}{\sigma_\varepsilon^2}\tfrac{\sum_{j=1}^{J_i}(y_{ij}-\mu-\beta_{j(i)})}{\frac{1}{\sigma_\alpha^2}+\frac{J_i}{\sigma_\varepsilon^2}}]^2]\\
&\quad\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2}+\tfrac{J_i}{\sigma_\varepsilon^2}][\alpha_i-J_i\tfrac{1}{\sigma_\varepsilon^2}\tfrac{y_{i\cdot}-\mu-\frac{1}{J_i}\sum_{j=1}^{J_i}\beta_{j(i)}}{\frac{1}{\sigma_\alpha^2}+\frac{J_i}{\sigma_\varepsilon^2}}]^2]
\end{aligned}
$$

giving the second result. Finally,

$$
\begin{aligned}
&p(\beta_{j(i)}|\mu, \boldsymbol{\alpha}_{-i}, \boldsymbol{\beta}, \boldsymbol{y})\\
&\quad \propto p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{y})\\
&\quad = p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\boldsymbol{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})\\
&\quad \propto p(\beta_{j(i)})p(\boldsymbol{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})\\
&\quad \propto \exp(-\tfrac{1}{2\sigma_\beta^2}\beta_{j(i)}^2)\exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij}-\mu-\alpha_i-\beta_{j(i)})^2]\\
&\quad \propto \exp(-\tfrac{1}{2\sigma_\beta^2}\beta_{j(i)}^2)\exp[-\tfrac{1}{2\sigma_\varepsilon^2}[(y_{ij}-\mu-\alpha_i)^2-2(y_{ij}-\mu-\alpha_i)\beta_{j(i)}+\beta_{j(i)}^2]]\\
&\quad \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2}\beta_{j(i)}^2+\tfrac{1}{\sigma_\varepsilon^2}\beta_{j(i)}^2-2\tfrac{1}{\sigma_\varepsilon^2}(y_{ij}-\mu-\alpha_i)\beta_{j(i)}]\\
&\quad \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2}+\tfrac{1}{\sigma_\varepsilon^2}][\beta_{j(i)}-\tfrac{1}{\sigma_\varepsilon^2}\tfrac{(y_{ij}-\mu-\alpha_i)}{\frac{1}{\sigma_\beta^2}+\frac{1}{\sigma_\varepsilon^2}}]^2]\\
&\quad \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2}+\tfrac{1}{\sigma_\varepsilon^2}][\beta_{j(i)}-\tfrac{1}{\sigma_\varepsilon^2}\tfrac{y_{ij}-\mu-\alpha_i)}{\frac{1}{\sigma_\alpha^2}+\frac{1}{\sigma_\varepsilon^2}}]^2]
\end{aligned}
$$

(b). The now model is now

$$
\begin{aligned}
Y_{ij} &= \eta_{ij} + \varepsilon_{ij}\\
\eta_{ij} &\sim N(\gamma_i, \sigma_\beta^2)\\
\gamma_i &\sim N(\mu, \sigma_\alpha^2)\\
f(\mu) &\propto 1
\end{aligned}
$$

We then have

$$
\begin{aligned}
p(\mu|\boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) &\propto p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) = p(\mu)p(\boldsymbol{\gamma}|\boldsymbol{\mu})p(\boldsymbol{\eta}|\gamma)p(\boldsymbol{y}|\boldsymbol{\eta})\\
&\propto p(\mu)p(\boldsymbol{\gamma}|\mu)\\
&\propto \prod_{i=1}^{I}\exp[-\tfrac{1}{2\sigma_\alpha^2}(\gamma_i-\mu)^2] = \exp[-\tfrac{1}{2\sigma_\alpha^2}[I\mu^2-2\sum_{i=1}^{I}\gamma_i^2\mu]\\
&\propto \exp[-\tfrac{I}{2\sigma_\alpha^2}[\mu-\tfrac{1}{I}\sum_{i=1}^{I}J_i\gamma_i]^2]
\end{aligned}
$$

Further,

$$\begin{aligned}
p(\gamma_i|\mu, \boldsymbol{\gamma}_{-i}, \boldsymbol{\beta}, \boldsymbol{y}) \propto & p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) \\
= & p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\boldsymbol{\gamma})p(\boldsymbol{y}|\boldsymbol{\eta}) \propto (\gamma_i|\mu)p(\boldsymbol{\eta}_i|\gamma_i) \\
& \propto \exp(-\tfrac{1}{2\sigma_\alpha^2}(\gamma_i - \mu)^2) \prod_{j=1}^{J_i} \exp[-\tfrac{1}{2\sigma_\beta^2}(\eta_{ij} - \gamma_i)^2] \\
& \propto \exp[-\tfrac{1}{2}[(\tfrac{1}{\sigma_\alpha^2} + \tfrac{J_i}{\sigma_\beta^2})\gamma_i^2 - 2(\tfrac{1}{\sigma_\alpha^2}\mu + \tfrac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij})\gamma_i] \\
& \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2} + \tfrac{J_i}{\sigma_\beta^2}][\gamma_i - \tfrac{\frac{1}{\sigma_\alpha^2}\mu + \frac{J_i}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij})}{\frac{1}{\sigma_\alpha^2} + \frac{1}{\sigma_\beta^2}}]^2]
\end{aligned}$$

and finally

$$\begin{aligned}
p(\eta_{ij}|\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}_{-ij}\boldsymbol{y}) \propto & p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) \\
= & p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\gamma)p(\boldsymbol{y}|\boldsymbol{\eta}) \propto p(\eta_{ij})p(y_{ij}|\eta_{ij}) \\
& \propto \exp(-\tfrac{1}{2\sigma_\beta^2}(\eta_{ij} - \gamma_i)^2) \exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij} - \eta_{ij})^2] \\
& \propto \exp[-\tfrac{1}{2}[(\tfrac{1}{\sigma_\beta^2} + \tfrac{1}{2\sigma_\varepsilon^2})\eta_{ij)}^2 - 2(\tfrac{1}{\sigma_\beta^2}\gamma_i + \tfrac{1}{\sigma_\varepsilon^2}y_{ij})\eta_{ij}] \\
& \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2} + \tfrac{1}{\sigma_\varepsilon^2}][\eta_{ij} - \tfrac{\frac{1}{\sigma_\beta^2}\gamma_i + \frac{1}{\sigma_\varepsilon^2}y_{ij}}{\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}}]^2] \\
& \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2} + \tfrac{1}{\sigma_\varepsilon^2}][\beta_{j(i)} - \tfrac{1}{\sigma_\varepsilon^2}\tfrac{y_{ij} - \mu - \alpha_i)}{\frac{1}{\sigma_\alpha^2} + \frac{1}{\sigma_\varepsilon^2}}]^2]
\end{aligned}$$