UiO : Matematisk institutt

Det matematisk-naturvitenskapelige fakultet

STK-4051/9051 Computational Statistics Spring 2024
Exercise 11

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7.6. Problem 6.4 introduces data on coal-mining disasters from 1851 to 1962. For these data, assume the model

$$X_j \sim \begin{cases} \text{Poisson}(\lambda_1), & j = 1, \ldots, \theta, \\ \text{Poisson}(\lambda_2), & j = \theta + 1, \ldots, 112. \end{cases} \quad (7.28)$$

Assume $\lambda_i|\alpha \sim \text{Gamma}(3, \alpha)$ for $i = 1, 2$, where $\alpha \sim \text{Gamma}(10, 10)$, and assume θ follows a discrete uniform distribution over $\{1, \ldots, 111\}$. The goal of this problem is to estimate the posterior distribution of the model parameters via a Gibbs sampler.

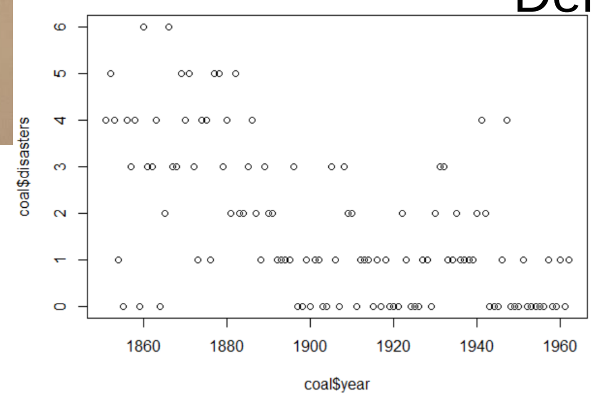
a. Derive the conditional distributions necessary to carry out Gibbs sampling for the change-point model.

b. Implement the Gibbs sampler. Use a suite of convergence diagnostics to evaluate the convergence and mixing of your sampler.

c. Construct density histograms and a table of summary statistics for the approximate posterior distributions of θ, λ_1, and λ_2. Are symmetric HPD intervals appropriate for all of these parameters?

d. Interpret the results in the context of the problem.

Higest Posterior Density



Parameters $\theta, \lambda_1, \lambda_2, \alpha$

Exersice 7.6

$$p(\alpha,\theta,\lambda_1,\lambda_2|x) \propto p(\alpha)p(\theta)\cdot p(\lambda_1|\alpha\,)\cdot p(\lambda_2|\alpha)\times L(\theta,\lambda_1,\lambda_2)$$

$$\frac{10^{10}\alpha^{10-1}}{\Gamma(10)}e^{-10\alpha}\frac{1}{111}\ \frac{\alpha^3\lambda_1^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_1}\frac{\alpha^3\lambda_2^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_2}\times\prod_{j=1}^{\theta}\frac{\lambda_1^{x_j\ -\lambda_1}}{e}\quad x_j!\prod_{j=\theta+1}^{112}\frac{\lambda_2^{x_j}e^{-\lambda_2}}{x_j!}$$

$$p(\lambda_1,\lambda_2|\cdots)\propto\frac{\alpha^3\lambda_1^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_1}\frac{\alpha^3\lambda_2^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_2}\times$$

$$\prod_{j=1}^{\theta}\frac{\lambda_1^{x_j}e^{-\lambda_1}}{x_j!}\prod_{j=\theta+1}^{112}\frac{\lambda_2^{x_j}e^{-\lambda_2}}{x_j!}$$

$$\propto\lambda_1^{2+\sum_{j=1}^{\theta}x_j}e^{-(\alpha+\theta)\lambda_1}\lambda_2^{2+\sum_{j=\theta+1}^{112}x_j}e^{-(\alpha+112-\theta)\lambda_2}$$

$$\propto\text{Gamma}(\lambda_1;3+\sum_{j=1}^{\theta}x_j,\alpha+\theta)\text{Gamma}(\lambda_2;3+\sum_{j=\theta+1}^{112}x_j,\alpha+112-\theta)$$

Exersice 7.6

$$p(\theta,\lambda_1,\lambda_2) \propto p(\alpha)p(\theta)\cdot p(\lambda_1|\alpha)\cdot p(\lambda_2|\alpha)\times L(\theta,\lambda_1,\lambda_2)$$

$$\frac{10^{10}\alpha^{10-1}}{\Gamma(10)}e^{-10\alpha}\frac{1}{111}\ \frac{\alpha^3\lambda_1^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_1}\frac{\alpha^3\lambda_2^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_2}\times\prod_{j=1}^{\theta}\frac{\lambda_1^{x_j}{}^{-\lambda_1}}{e}x_j!\prod_{j=\theta+1}^{112}\frac{\lambda_2^{x_j}e^{-\lambda_2}}{x_j!}$$

$$\begin{aligned}p(\alpha|\cdots) &\propto \frac{10^{10}\alpha^{10-1}}{\Gamma(10)}e^{-10\alpha}\frac{\alpha^3\lambda_1^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_1}\frac{\alpha^3\lambda_2^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_2}\\ &\propto \alpha^{15}e^{-(10+\lambda_1+\lambda_2)\alpha}\\ &\propto \text{Gamma}(\alpha;16,10+\lambda_1+\lambda_2)\end{aligned}$$

Exersice 7.6

$$p(\theta,\lambda_1,\lambda_2\ ) \propto p(\alpha)p(\theta)\cdot p(\lambda_1|\alpha\ )\cdot p(\lambda_2|\alpha) \times L(\theta,\lambda_1,\lambda_2)$$

$$\frac{10^{10}\alpha^{10-1}}{\Gamma(10)}e^{-10\alpha}\frac{1}{111}\ \frac{\alpha^3\lambda_1^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_1}\frac{\alpha^3\lambda_2^{3-1}}{\Gamma(3)}e^{-\alpha\lambda_2}\times\prod_{j=1}^{\theta}\frac{\lambda_1^{x_j\ -\lambda_1}}{e}\ x_j!\prod_{j=\theta+1}^{112}\frac{\lambda_2^{x_j}e^{-\lambda_2}}{x_j!}$$

$$p(\theta|\ldots)\ \propto\frac{1}{111}\prod_{j=1}^{\theta}\frac{\lambda_1^{x_j\ -\lambda_1}}{e}\ x_j!\prod_{j=\theta+1}^{112}\frac{\lambda_2^{x_j}e^{-\lambda_2}}{x_j!}$$

$$\propto\lambda_1^{\sum_{j=1}^{\theta}x_j}e^{-\theta\lambda_1}\lambda_2^{\sum_{j=\theta+1}^{112}x_j}e^{-\theta\lambda_2}$$

Not a known distribution, But a finite number of values for theta, we need the partial sum of x_j

```
#Calculating sum_xi for all values of theta
xsum = rep(NA,n-1)
for(i in 1:(n-1))
  xsum[i] = sum(coal$disasters[1:i])
Xsum = sum(coal$disasters)
```

7.6 b

```
for(i in 1:N)
{
  #Sample theta
  logp = xsum*log(lambda1)-lambda1*seq(1,n-1) + (Xsum-xsum)*log(lambda2)-lambda2*seq(n-1,1)
  p = exp(logp)/sum(exp(logp))
  theta = sample(1:(n-1),1,prob=p)
  #Sample lambda
  lambda1 = rgamma(1,shape=3+xsum[theta],rate=alpha+theta)
  lambda2 = rgamma(1,shape=3+Xsum-xsum[theta],rate=alpha+n-theta)
  #Sample alpha
  alpha = rgamma(1,shape=16,rate=10+lambda1+lambda2)
  thetaM = c(thetaM,theta)
  lambdaM1 = c(lambdaM1,lambda1)
  lambdaM2 = c(lambdaM2,lambda2)
  alphaM = c(alphaM,alpha)
}
```

$$\propto \lambda_1^{\sum_{j=1}^{\theta} x_j} e^{-\theta\lambda_1} \lambda_2^{\sum_{j=\theta+1}^{112} x_j} e^{-\theta\lambda_2}$$

$$\text{Gamma}(\lambda_1; 3 + \sum_{j=1}^{theta}, \alpha + \theta)$$

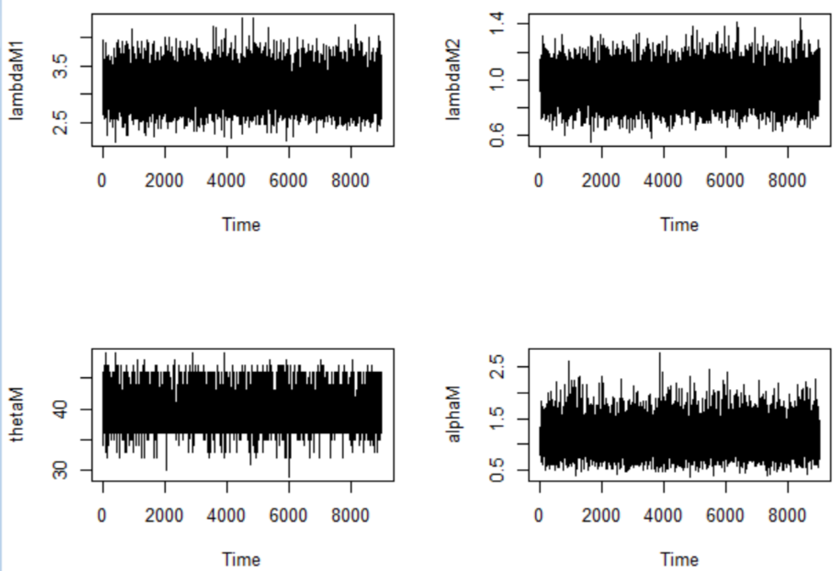
$$\text{Gamma}(\lambda_2; 3 + \sum_{j=\theta+1}^{112}, \alpha + 112 - \theta)$$

$$\propto \text{Gamma}(\alpha; 16, 10 + \lambda_1 + \lambda_2)$$

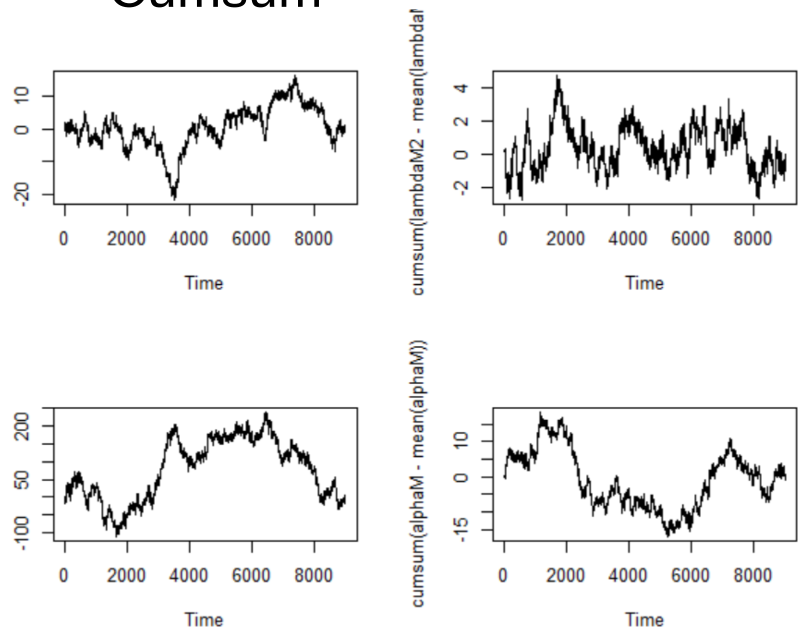
Diagnostics

Have removed the 1000 first

Trace plot

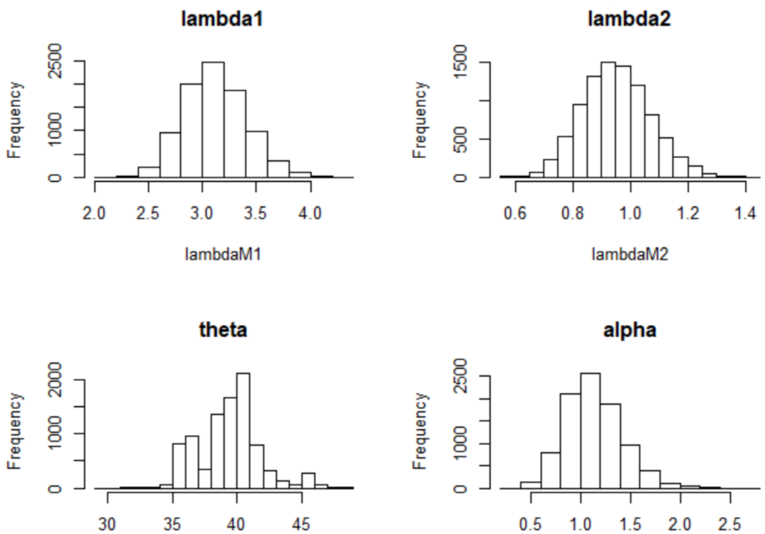


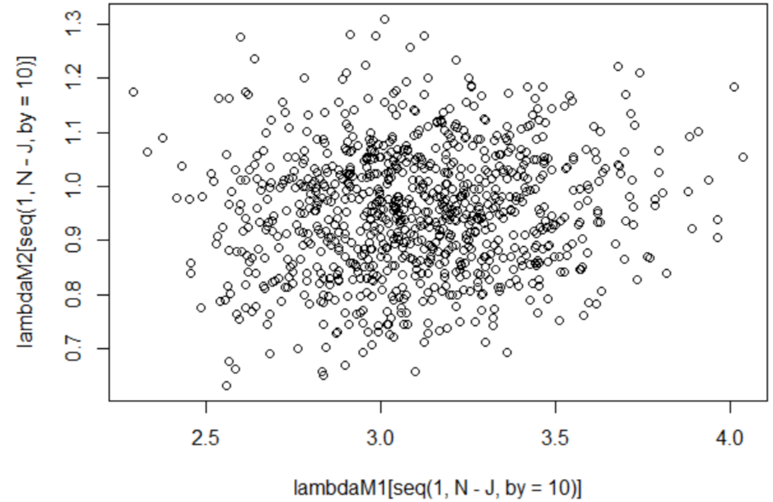
Cumsum

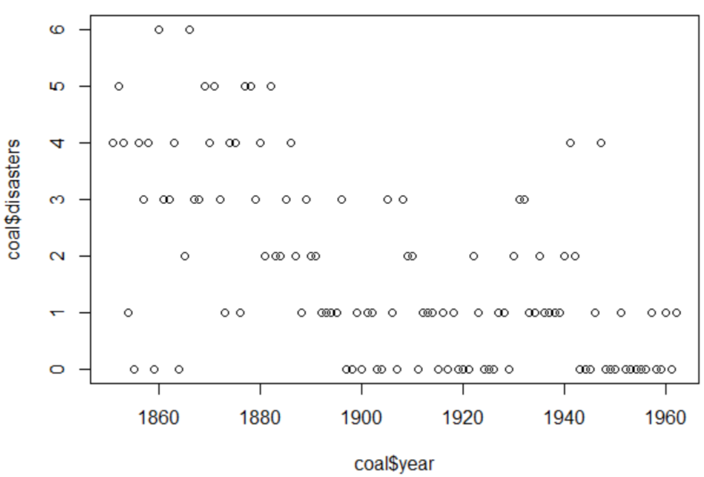


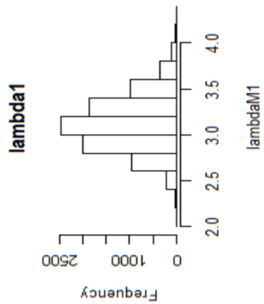
Results

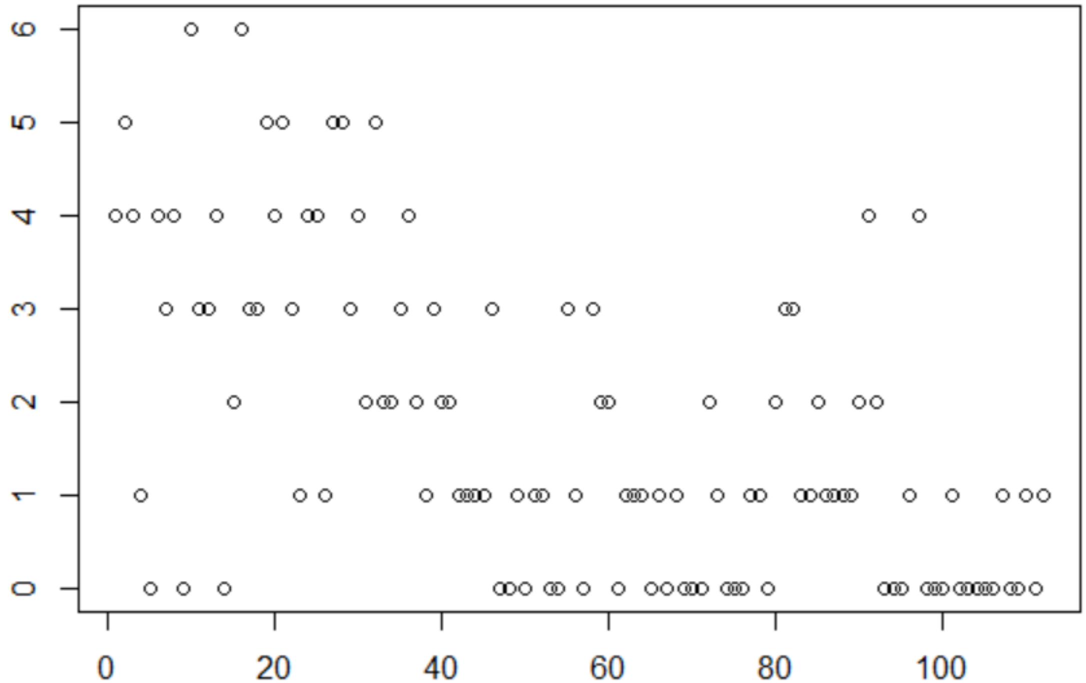
```
par(mfrow=c(1,1))
plot(lambdaM1[seq(1,N-J,by=10)],lambdaM2[seq(1,N-J,by=10)])
```

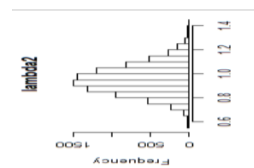


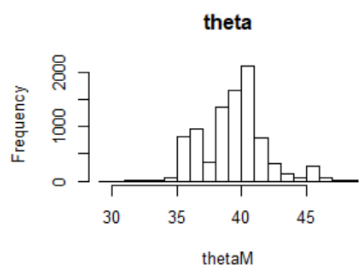




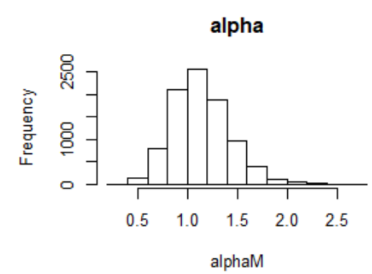

lambda1
Frequency
lambdaM1




lambda2
Frequency


theta
Frequency
thetaM

850


alpha
Frequency
alphaM

7.7. Consider a hierarchical nested model

$$Y_{ijk} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ijk}, \tag{7.29}$$

where $i = 1, \ldots, I$, $j = 1, \ldots, J_i$, and $k = 1, \ldots, K$. After averaging over k for each i and j, we can rewrite the model (7.29) as

$$Y_{ij} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ij}, \qquad i = 1, \ldots, I, \quad j = 1, \ldots, J_i, \tag{7.30}$$

where $Y_{ij} = \sum_{k=1}^{K} Y_{ijk}/K$. Assume that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$, and $\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$, where each set of parameters is independent a priori. Assume that σ_α^2, σ_β^2, and σ_ϵ^2 are known. To carry out Bayesian inference for this model, assume an improper flat prior for μ, so $f(\mu) \propto 1$. We consider two forms of the Gibbs sampler for this problem [546]:

$$Y_{ij} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ij}, \qquad i = 1, \ldots, I, \quad j = 1, \ldots, J_i, \tag{7.30}$$

where $Y_{ij} = \sum_{k=1}^{K} Y_{ijk}/K$. Assume that $\alpha_i \sim N(0, \sigma_\alpha^2)$, $\beta_{j(i)} \sim N(0, \sigma_\beta^2)$, and $\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$, where each set of parameters is independent a priori. Assume that σ_α^2, σ_β^2, and σ_ϵ^2 are known. To carry out Bayesian inference for this model, assume an improper flat prior for μ, so $f(\mu) \propto 1$. We consider two forms of the Gibbs sampler for this problem [546]:

a. Let $n = \sum_i J_i$, $y_{\cdot\cdot} = \sum_{ij} y_{ij}/n$, and $y_{i\cdot} = \sum_j y_{ij}/J_i$ hereafter. Show that at iteration t, the conditional distributions necessary to carry out Gibbs sampling for this

$$\mu^{(t+1)} \big| (\boldsymbol{\alpha}^{(t)}, \boldsymbol{\beta}^{(t)}, \mathbf{y}) \sim N\left(y_{\cdot\cdot} - \frac{1}{n} \sum_i J_i \alpha_i^{(t)} - \frac{1}{n} \sum_{j(i)} \beta_{j(i)}^{(t)}, \frac{\sigma_\epsilon^2}{n} \right),$$

$$\alpha_i^{(t+1)} \big| (\mu^{(t+1)}, \boldsymbol{\beta}^{(t)}, \mathbf{y}) \sim N\left(\frac{J_i V_1}{\sigma_\epsilon^2} \left(y_{i\cdot} - \mu^{(t+1)} - \frac{1}{J_i} \sum_j \beta_{j(i)}^{(t)} \right), V_1 \right),$$

$$\beta_{j(i)}^{(t+1)} \big| (\mu^{(t+1)}, \boldsymbol{\alpha}^{(t+1)}, \mathbf{y}) \sim N\left(\frac{V_2}{\sigma_\epsilon^2} \left(y_{ij} - \mu^{(t+1)} - \alpha_i^{(t+1)} \right), V_2 \right),$$

where

$$V_1 = \left(\frac{J_i}{\sigma_\epsilon^2} + \frac{1}{\sigma_\alpha^2} \right)^{-1} \quad \text{and} \quad V_2 = \left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\beta^2} \right)^{-1}.$$

$$\phi(z; \text{Mean}, \text{Var}) \propto \exp\{-\frac{1}{2} \cdot \frac{1}{\text{Var}} \ (z - \text{Mean})^2 \ \}$$

$$n\left(\mu - \frac{1}{n}x\right)^2 = (n\mu^2 - 2\mu x + ..)$$

$$
\begin{aligned}
& p(\mu|\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{y}) \\
& \quad \propto p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{y}) \\
& \quad = p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\boldsymbol{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \quad \propto p(\mu)p(\boldsymbol{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) \\
& \quad \propto \prod_{i=1}^{I}\prod_{j=1}^{J_i} \exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i - \beta_{j(i)})^2] \\
& \quad \propto \prod_{i=1}^{I}\prod_{j=1}^{J_i} \exp[-\tfrac{1}{2\sigma_\varepsilon^2}[(y_{ij} - \alpha_i - \beta_{j(i)})^2 - 2(y_{ij} - \alpha_i - \beta_{j(i)})\mu + \mu^2]] \\
& \quad \propto \exp[-\tfrac{1}{2\sigma_\varepsilon^2}[n\mu^2 - 2\sum_{i=1}^{I}\sum_{j=1}^{J_i}(y_{ij} - \alpha_i - \beta_{j(i)})\mu] \\
& \quad \propto \exp[-\tfrac{n}{2\sigma_\varepsilon^2}[\mu - \tfrac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}(y_{ij} - \alpha_i - \beta_{j(i)})]^2] \\
& \quad \propto \exp[-\tfrac{n}{2\sigma_\varepsilon^2}[\mu - (y_{..} - \tfrac{1}{n}\sum_{i=1}^{I}J_i\alpha_i - \tfrac{1}{n}\sum_{i=1}^{I}\sum_{j=1}^{J_i}\beta_{j(i)})]^2]
\end{aligned}
$$

$$c_1\left(\alpha-\frac{1}{c_1}x\right)^2 = (c_1\alpha^{\,2}-2\alpha x+..)$$

$$
\begin{aligned}
&p(\alpha_i|\mu,\boldsymbol{\alpha}_{-i},\boldsymbol{\beta},\boldsymbol{y})\\
&\quad\propto p(\mu,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{y})\\
&\quad= p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\boldsymbol{y}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\quad\propto p(\alpha_i)p(\boldsymbol{y}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})\\
&\quad\propto \exp(-\tfrac{1}{2\sigma_\alpha^2}\alpha_i^2)\prod_{j=1}^{J_i}\exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij}-\mu-\alpha_i-\beta_{j(i)})^2]\\
&\quad\propto \exp(-\tfrac{1}{2\sigma_\alpha^2}\alpha_i^2)\prod_{j=1}^{J_i}\exp[-\tfrac{1}{2\sigma_\varepsilon^2}[(y_{ij}-\mu-\beta_{j(i)})^2-2(y_{ij}-\mu-\beta_{j(i)})\alpha_i+\alpha_i^2]]\\
&\quad\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2}\alpha_i^2+\tfrac{1}{\sigma_\varepsilon^2}J_i\alpha_i^2-2\tfrac{1}{\sigma_\varepsilon^2}\sum_{j=1}^{J_i}(y_{ij}-\mu-\beta_{j(i)})\alpha_i]\\
&\quad\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2}+\tfrac{J_i}{\sigma_\varepsilon^2}][\alpha_i-\tfrac{1}{\sigma_\varepsilon^2}\frac{\sum_{j=1}^{J_i}(y_{ij}-\mu-\beta_{j(i)})}{\frac{1}{\sigma_\alpha^2}+\frac{J_i}{\sigma_\varepsilon^2}}]^2]\\
&\quad\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2}+\tfrac{J_i}{\sigma_\varepsilon^2}][\alpha_i-J_i\tfrac{1}{\sigma_\varepsilon^2}\frac{y_{i\cdot}-\mu-\frac{1}{J_i}\sum_{j=1}^{J_i}\beta_{j(i)})}{\frac{1}{\sigma_\alpha^2}+\frac{J_i}{\sigma_\varepsilon^2}}]^2]
\end{aligned}
$$

$$c_1\left(\beta - \frac{1}{c_1}x\right)^2 = (c_1\beta^{\,2} - 2\beta x + ..)$$

$$
\begin{aligned}
&p(\beta_{j(i)}|\mu, \boldsymbol{\alpha}_{-i}, \boldsymbol{\beta}, \boldsymbol{y})\\
&\quad \propto p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{y})\\
&\quad = p(\mu)p(\boldsymbol{\alpha})p(\boldsymbol{\beta})p(\boldsymbol{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})\\
&\quad \propto p(\beta_{j(i)})p(\boldsymbol{y}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})\\
&\quad \propto \exp(-\tfrac{1}{2\sigma_\beta^2}\beta_{j(i)}^2)\exp[-\tfrac{1}{2\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i - \beta_{j(i)})^2]\\
&\quad \propto \exp(-\tfrac{1}{2\sigma_\beta^2}\beta_{j(i)}^2)\exp[-\tfrac{1}{2\sigma_\varepsilon^2}[(y_{ij} - \mu - \alpha_i)^2 - 2(y_{ij} - \mu - \alpha_i)\beta_{j(i)} + \beta_{j(i)}^2]]\\
&\quad \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2}\beta_{j(i)}^2 + \tfrac{1}{\sigma_\varepsilon^2}\beta_{j(i)}^2 - 2\tfrac{1}{\sigma_\varepsilon^2}(y_{ij} - \mu - \alpha_i)\beta_{j(i)}]\\
&\quad \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2} + \tfrac{1}{\sigma_\varepsilon^2}][\beta_{j(i)} - \tfrac{1}{\sigma_\varepsilon^2}\tfrac{(y_{ij}-\mu-\alpha_i)}{\frac{1}{\sigma_\beta^2}+\frac{1}{\sigma_\varepsilon^2}}]^2]\\
&\quad \propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\beta^2} + \tfrac{1}{\sigma_\varepsilon^2}][\beta_{j(i)} - \tfrac{1}{\sigma_\varepsilon^2}\tfrac{y_{ij}-\mu-\alpha_i)}{\frac{1}{\sigma_\alpha^2}+\frac{1}{\sigma_\varepsilon^2}}]^2]
\end{aligned}
$$

b. The convergence rate for a Gibbs sampler can sometimes be improved via reparameterization. For this model, the model can be reparameterized via hierarchical centering (Section 7.3.1.4). For example, let Y_{ij} follow (7.30), but now let $\eta_{ij} = \mu + \alpha_i + \beta_{j(i)}$ and $\epsilon_{ij} \sim N\left(0, \sigma_\epsilon^2\right)$. Then let $\gamma_i = \mu + \alpha_i$ with $\eta_{ij}|\gamma_i \sim N\left(\gamma_i, \sigma_\beta^2\right)$ and $\gamma_i|\mu \sim N\left(\mu, \sigma_\alpha^2\right)$. As above, assume σ_α^2, σ_β^2, and σ_ϵ^2 are known, and assume a flat prior for μ. Show that the conditional distributions necessary to carry out Gibbs sampling for this model are given by

$$\mu^{(t+1)} \big| \left(\boldsymbol{\gamma}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbf{y}\right) \sim N\left(\frac{1}{I}\sum_i \gamma_i^{(t)}, \frac{1}{I}\sigma_\alpha^2\right),$$

$$\gamma_i^{(t+1)} \big| \left(\mu^{(t+1)}, \boldsymbol{\eta}^{(t)}, \mathbf{y}\right) \sim N\left(V_3\left(\frac{1}{\sigma_\beta^2}\sum_j \eta_{ij}^{(t)} + \frac{\mu^{(t+1)}}{\sigma_\alpha^2}\right), V_3\right),$$

$$\eta_{ij}^{(t+1)} \big| \left(\mu^{(t+1)}, \boldsymbol{\gamma}^{(t+1)}, \mathbf{y}\right) \sim N\left(V_2\left(\frac{y_{ij}}{\sigma_\epsilon^2} + \frac{\gamma_i^{(t+1)}}{\sigma_\beta^2}\right), V_2\right),$$

where

$$V_3 = \left(\frac{J_i}{\sigma_\beta^2} + \frac{1}{\sigma_\alpha^2}\right)^{-1}.$$

$$Y_{ij} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ij}, \qquad i = 1, \ldots, I, \quad j = 1, \ldots, J_i,$$

$$\begin{aligned} Y_{ij} &= \eta_{ij} + \varepsilon_{ij} \\ \eta_{ij} &\sim N(\gamma_i, \sigma_\beta^2) \\ \gamma_i &\sim N(\mu, \sigma_\alpha^2) \\ f(\mu) &\propto 1 \end{aligned}$$

$$\begin{aligned} p(\mu|\boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) &\propto p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) = p(\mu)p(\boldsymbol{\gamma}|\boldsymbol{\mu})p(\boldsymbol{\eta}|\gamma)p(\boldsymbol{y}|\boldsymbol{\eta}) \\ &\propto p(\mu)p(\boldsymbol{\gamma}|\mu) \\ &\propto \prod_{i=1}^{I} \exp[-\tfrac{1}{2\sigma_\alpha^2}(\gamma_i - \mu)^2] = \exp[-\tfrac{1}{2\sigma_\alpha^2}[I\mu^2 - 2\sum_{i=1}^{I} \gamma_i^2 \mu] \\ &\propto \exp[-\tfrac{I}{2\sigma_\alpha^2}[\mu - \tfrac{1}{I}\sum_{i=1}^{I} \gamma_i]^2] \end{aligned}$$

$$
\begin{aligned}
Y_{ij} &= \eta_{ij} + \varepsilon_{ij} \\
\eta_{ij} &\sim N(\gamma_i, \sigma_\beta^2) \\
\gamma_i &\sim N(\mu, \sigma_\alpha^2) \\
f(\mu) &\propto 1
\end{aligned}
$$

$$
\begin{aligned}
p(\gamma_i|\mu, \boldsymbol{\gamma}_{-i}, \boldsymbol{\beta}, \boldsymbol{y}) &\propto p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) \\
&= p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\boldsymbol{\gamma})p(\boldsymbol{y}|\boldsymbol{\eta}) \propto p(\gamma_i|\mu)p(\boldsymbol{\eta}_i|\gamma_i) \\
&\propto \exp(-\tfrac{1}{2\sigma_\alpha^2}(\gamma_i - \mu)^2) \prod_{j=1}^{J_i} \exp[-\tfrac{1}{2\sigma_\beta^2}(\eta_{ij} - \gamma_i)^2] \\
&\propto \exp[-\tfrac{1}{2}[(\tfrac{1}{\sigma_\alpha^2} + \tfrac{J_i}{\sigma_\beta^2})\gamma_i^2 - 2(\tfrac{1}{\sigma_\alpha^2}\mu + \tfrac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij})\gamma_i] \\
&\propto \exp[-\tfrac{1}{2}[\tfrac{1}{\sigma_\alpha^2} + \tfrac{J_i}{\sigma_\beta^2}][\gamma_i - \frac{\frac{1}{\sigma_\alpha^2}\mu + \frac{1}{\sigma_\beta^2}\sum_{j=1}^{J_i}\eta_{ij})}{\frac{1}{\sigma_\alpha^2} + \frac{J_i}{\sigma_\beta^2}}]^2]
\end{aligned}
$$

$$\begin{aligned} Y_{ij} =& \eta_{ij} + \varepsilon_{ij} \\ \eta_{ij} \sim& N(\gamma_i, \sigma_\beta^2) \\ \gamma_i \sim& N(\mu, \sigma_\alpha^2) \\ f(\mu) \propto& 1 \end{aligned}$$

$$\begin{aligned} p(\eta_{ij}|\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}_{-ij}\boldsymbol{y}) \propto& p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \boldsymbol{y}) \\ =& p(\mu)p(\boldsymbol{\gamma}|\mu)p(\boldsymbol{\eta}|\gamma)p(\boldsymbol{y}|\boldsymbol{\eta}) \propto p(\eta_{ij})p(y_{ij}|\eta_{ij}) \\ &\propto \exp(-\frac{1}{2\sigma_\beta^2}(\eta_{ij} - \gamma_i)^2) \exp[-\frac{1}{2\sigma_\varepsilon^2}(y_{ij} - \eta_{ij})^2] \\ &\propto \exp[-\frac{1}{2}[(\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2})\eta_{ij}^2 \; - 2(\frac{1}{\sigma_\beta^2}\gamma_i + \frac{1}{\sigma_\varepsilon^2}y_{ij})\eta_{ij}] \\ &\propto \exp[-\frac{1}{2}[\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}][\eta_{ij} - \frac{\frac{1}{\sigma_\beta^2}\gamma_i + \frac{1}{\sigma_\varepsilon^2}y_{ij}}{\frac{1}{\sigma_\beta^2} + \frac{1}{\sigma_\varepsilon^2}}]^2] \end{aligned}$$

7.8 a Gibbs for $p(\mu, \alpha, \beta, y)$

- Initialization

```
sig2.alpha = 86
sig2.beta = 58
sig2.eps = 1
n = nrow(d)
y.bar = mean(d$Moisture)
y.dot <-with(d, tapply(Moisture, Batch, mean))
J = 2
I = 15
D=1000
L = 1000
N = D+L
V1 = 1/(J/sig2.eps + 1/sig2.alpha)
V2 = 1/(1/sig2.eps+1/sig2.beta)
V3 = 1/(J/sig2.beta+1/sig2.alpha)
y = matrix(d$Moisture,ncol=J,byrow=T)
```

Given in text (sig2.alpha, sig2.beta, sig2.eps)

$$V_1 = \left(\frac{J_i}{\sigma_\epsilon^2} + \frac{1}{\sigma_\alpha^2}\right)^{-1} \quad \text{and} \quad V_2 = \left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\beta^2}\right)^{-1}$$

$$V_3 = \left(\frac{J_i}{\sigma_\beta^2} + \frac{1}{\sigma_\alpha^2}\right)^{-1}$$

7.8 a Gibbs for $p(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{y})$

- Gibbs sampler

```
#Gibbs sampling
for(i in 1:N)
{
  #Sample mu
  mu = rnorm(1,y.bar-sum(J*alpha)/n-sum(beta)/n,sqrt(sig2.eps)/n)
    #Sample alpha
  alpha = rnorm(I,J*V1*(y.dot-mu-rowSums(beta)/J)/sig2.eps,sqrt(V1))
  #Sample beta
  beta[,1] = rnorm(I,V2*(y[,1]-mu-alpha),sqrt(V2))
  beta[,2] = rnorm(I,V2*(y[,2]-mu-alpha),sqrt(V2))
  #Store simulations
  muM1[k,i] = mu
  alphaM[k,i,] = alpha
  betaM[k,i,,] = beta
}
```

$$\mu^{(t+1)} \mid (\boldsymbol{\alpha}^{(t)}, \boldsymbol{\beta}^{(t)}, \mathbf{y}) \sim N\left(y_{\cdot\cdot} - \frac{1}{n}\sum_i J_i \alpha_i^{(t)} - \frac{1}{n}\sum_{j(i)} \beta_{j(i)}^{(t)}, \frac{\sigma_\epsilon^2}{n}\right),$$

$$\alpha_i^{(t+1)} \mid (\mu^{(t+1)}, \boldsymbol{\beta}^{(t)}, \mathbf{y}) \sim N\left(\frac{J_i V_1}{\sigma_\epsilon^2}\left(y_{i\cdot} - \mu^{(t+1)} - \frac{1}{J_i}\sum_j \beta_{j(i)}^{(t)}\right), V_1\right)$$

$$\beta_{j(i)}^{(t+1)} \mid (\mu^{(t+1)}, \boldsymbol{\alpha}^{(t+1)}, \mathbf{y}) \sim N\left(\frac{V_2}{\sigma_\epsilon^2}\left(y_{ij} - \mu^{(t+1)} - \alpha_i^{(t+1)}\right), V_2\right),$$

```
sig2.eps = 1
```

7.8 b) Gibbs for $p(\mu, \boldsymbol{\gamma}, \boldsymbol{\eta}, \mathbf{y})$

```
#Initialization
mu = y.bar
gamma = y.dot-y.bar
eta = matrix(0,nrow=I,ncol=J)
y = matrix(d$Moisture,ncol=J,byrow=T)
#Gibbs sampling
for(i in 1:N)
{
  #Sample mu
  mu = rnorm(1,mean(gamma),sqrt(sig2.alpha)/I)
  #Sample gamma
  gamma = rnorm(I,V3*(rowSums(eta)/sig2.beta + mu/sig2.alpha),sqrt(V3))
  #Sample eta
  eta[,1] = rnorm(I,V2*(y[,1]/sig2.eps-+gamma/sig2.beta),sqrt(V2))
  eta[,2] = rnorm(I,V2*(y[,2]/sig2.eps+gamma/sig2.beta),sqrt(V2))

  muM2[k,i] = mu
  gammaM[k,i,] = gamma
  etaM[k,i,,] = eta
}
```

$$\mu^{(t+1)} \big| \left(\boldsymbol{\gamma}^{(t)}, \boldsymbol{\eta}^{(t)}, \mathbf{y}\right) \sim N\left(\frac{1}{I}\sum \gamma_i^{(t)}, \frac{1}{I}\sigma_\alpha^2\right),$$

$$\gamma_i^{(t+1)} \big| \left(\mu^{(t+1)}, \boldsymbol{\eta}^{(t)}, \mathbf{y}\right) \sim N\left(V_3\left(\frac{1}{\sigma_\beta^2}\sum_j \eta_{ij}^{(t)} + \frac{\mu^{(t+1)}}{\sigma_\alpha^2}\right), V_3\right),$$

$$\eta_{ij}^{(t+1)} \big| \left(\mu^{(t+1)}, \boldsymbol{\gamma}^{(t+1)}, \mathbf{y}\right) \sim N\left(V_2\left(\frac{y_{ij}}{\sigma_\epsilon^2} + \frac{\gamma_i^{(t+1)}}{\sigma_\beta^2}\right), V_2\right),$$