Problem 1

a) See S&S page 92 and S&S page 94 and 95.

b) The characteristic polynomials equals

$$\phi(z) = 0.12z^2 - 0.1 - 1$$

which has roots

$$\frac{0.1 \pm \sqrt{0.01 + 4 \times 0.12}}{2 \times 0.12} = \begin{cases} 0.8/0.24 = 1/0.3 \\ -0.6/0.24 = -1/0.4 \end{cases}$$

Both are outside the unit circle so the time series is causal.

c) For definition of partial correlation function see S&S page 106. In a AR(p) model it has the property of being equal to zero for arguments larger than p. Hence from plotting the partial autocorrelation function one can determine the order from observing when the partial autocorrelation function is equal to zero.

Problem 2

(i) For the *trend and cycle* model let the state be

$$\mathbf{x_t} = \begin{pmatrix} \mu_t \\ \beta_t \\ \psi_t \\ \psi_t^* \end{pmatrix}$$

Then the transition or state equation is given by

$$\begin{aligned} \mathbf{x_t} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5\cos\lambda & 0.5\sin\lambda \\ 0 & 0 & -0.5\sin\lambda & 0.5\cos\lambda \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \beta_{t-1} \\ \psi_{t-1} \\ \psi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \eta_t \\ \zeta_t \\ \psi_t \\ \psi_t^* \end{pmatrix} \\ &= \Phi \mathbf{x_{t-1}} + \begin{pmatrix} \eta_t \\ \zeta_t \\ \psi_t \\ \psi_t^* \end{pmatrix} \end{aligned}$$

and the measurement or observation equation

$$y_t = (1,0,1,0)\begin{pmatrix}\mu_t\\ \beta_t\\ \psi_t\\ \psi_t^*\end{pmatrix} + \epsilon_t = \mu_t + \psi_t + \epsilon_t.$$

For the *cyclical trend* model let the state be the same. Now the state equation is

$$\begin{aligned}\mathbf{x_t} &= \begin{pmatrix}1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0.5\cos\lambda & 0.5\sin\lambda\\ 0 & 0 & -0.5\sin\lambda & 0.5\cos\lambda\end{pmatrix} + \begin{pmatrix}\mu_{t-1}\\ \beta_{t-1}\\ \psi_{t-1}\\ \psi_{t-1}^*\end{pmatrix} + \begin{pmatrix}\eta_t\\ \zeta_t\\ \psi_t\\ \psi_t^*\end{pmatrix}\\ &= \Phi\mathbf{x_{t-1}} + \begin{pmatrix}\eta_t\\ \zeta_t\\ \psi_t\\ \psi_t^*\end{pmatrix}\end{aligned}$$

and the measurement or observation equation

$$y_t = (1,0,0,0)\begin{pmatrix}\mu_t\\ \beta_t\\ \psi_t\\ \psi_t^*\end{pmatrix} + \epsilon_t = \mu_t + \epsilon_t.$$

b) For *trend and cycle* model

$$\begin{aligned}\Phi P_0^0\Phi' &= \Phi\Phi' = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0.5\cos\lambda & 0.5\sin\lambda\\ 0 & 0 & -0.5\sin\lambda & 0.5\cos\lambda\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0.5\cos\lambda & -0.5\sin\lambda\\ 0 & 0 & 0.5\sin\lambda & 0.5\cos\lambda\end{pmatrix}\\ &= \begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0.25(\cos^2\lambda + \sin^2\lambda) & 0.25(-\cos\lambda\sin\lambda + \cos\lambda\sin\lambda)\\ 0 & 0 & 0.25(-\cos\lambda\sin\lambda + \cos\lambda\sin\lambda) & 0.25(\cos^2\lambda + \sin^2\lambda)\end{pmatrix}\\ &= \begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0.25 & 0\\ 0 & 0 & 0 & 0.25\end{pmatrix}\end{aligned}$$

so

$$P_1^0 = \Phi P_0^0 \Phi' + Q = + \begin{pmatrix} 2+\sigma_\eta^2 & 1 & 0 & 0 \\ 1 & 1+\sigma_\zeta^2 & 0 & 0 \\ 0 & 0 & 0.25+\sigma_\psi^2 & 0 \\ 0 & 0 & 0 & 0.25+\sigma_{\psi^*}^2 \end{pmatrix}$$

so $A\Phi P_1^0 A' + R = (1,0,1,0)(\Phi P_0^0 \Phi' + Q)(1,0,1,0)' + \sigma_\epsilon^2) = 2.25 + \sigma_\eta^2 + \sigma_\psi^2 + \sigma_\epsilon^2$. Since $P_1^0 A' = (2+\sigma_\eta^2, 1, 0.25+\sigma_\psi^2, 0)'$, $K_1 = P_1^0 A'(A\Phi P_1^0 A' + R)^{-1} = (2+\sigma_\eta^2, 1, 0.25+\sigma_\psi^2, 0)'/((2.25+\sigma_\eta^2+\sigma_\psi^2+\sigma_\epsilon^2)$

c) Using the Kalman filter y_t^{t-1} can be calculated for each $t = 1, \ldots, n$. But $y_t - y_t^{t-1}$ are independent $y_t - y_t^{t-1} \sim N(0, AP_t^{t-1}A' + \sigma_\epsilon^2)$. Hence the likelihood can be found as a function of σ_ϵ^2, σ_η^2,σ_ζ^2,σ_ψ^2 and $\sigma_{\psi^*}^2$. Using an optimization algorithm the maximal value can be found for each value of λ. Plotting the maxima as a function of λ, allows the maximum likelihood estimate for λ to be read off as the value that corresponds to the maximum value of the plot.

Problem 3

a) The periodogarm is the square of the modulus of the discrete Fourier transform, i.e. $I(j/n) = |d(j/n)|$, $j = 0, \ldots, n-1$. For large n $2I(j/n)/f_x(j/n)$, $j = 1, \ldots, [n/2]$ are approximately independent χ_2^2 variables. For parts of the specter where the spectral densities are not varying too much smoothed versions of the periodogram of the form $\frac{1}{(2m+1)} \sum_{k=-m}^{m} I((j+k)/n)$ will therefore have approximate expectation $f_x(j/n)$ and variance $f_x(j/n)^2/(2m+1)$.

b)

$$d_y(j/n) = \sum_{t=1}^{n} (\mu + x_t) e^{-2\pi i (j/n) t}/\sqrt{n} = \mu \sum_{t=1}^{n} e^{-2\pi i (j/n) t}/\sqrt{n} + \sum_{t=1}^{n} x_t e^{-2\pi i (j/n) t}/\sqrt{n}.$$

But

$$\begin{aligned} \sum_{t=1}^{n} e^{-2\pi i (j/n) t} &= e^{-2\pi i (j/n)} \sum_{t=0}^{n-1} e^{-2\pi i (j/n) t} \\ &= \begin{cases} n & \text{for } j = 0 \\ e^{-2\pi i (j/n)} \frac{1 - e^{-2\pi i (j/n) n}}{1 - e^{-2\pi i (j/n)}} = 0 & \text{for } j = 1, \ldots, n-1 \text{ and } j \neq n/2,\ n \text{ even} \\ -n & \text{for } j = n/2,\ n \text{ even.} \end{cases} \end{aligned}$$

Hence

$$d_y(j/n) = \begin{cases} \sqrt{n}(\mu + \bar{x}) & \text{for } j = 0 \\ d_x(j/n) & \text{for } j = 1, \ldots, n-1 \text{ and } j \neq n/2, \ n \text{ even} \\ -\sqrt{n}(\mu + \bar{x}) & \text{for } j = n/2, \ n \text{ even.} \end{cases}$$

cfr. S&S page 188

c) For $j = 1, \ldots, n'$ and $j \neq n'/2, \ n'$ even

$$\begin{aligned} d_{y'}(j/n') &= \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} \mu e^{-2\pi i(j/n')t} + \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} x_t e^{-2\pi i(j/n')t} \\ &= \frac{\mu}{\sqrt{n'}} e^{-2\pi i(j/n')} \sum_{t=0}^{n-1} e^{-2\pi i(j/n')(t-1)} + \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} x_t e^{-2\pi i(j/n')t} \\ &= \frac{\mu}{\sqrt{n'}} e^{-2\pi i(j/n')} \frac{1 - e^{-2\pi i(j/n')n}}{1 - e^{-2\pi i(j/n')}} + \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} x_t e^{-2\pi i(j/n')t}, \end{aligned}$$

for $j = 0$

$$d_{y'}(0) = (n\mu + n\bar{x})/\sqrt{n'}.$$

and for $j = n'/2, n' even$

$$d_{y'}(j/n') = -(n\mu + n\bar{x})/\sqrt{n'}.$$

d) The variables of interest are the observations $y_t, \ t = 1, \ldots, n$ whose Fourier transform only depends on μ for frequency 0. In the padded version the dependency of μ shows up in the discrete Fourier transform at all frequencies $j/n', \ j = 0, \ldots, n-1'$. Since padding is an auxiliary technique to facilitate the use of the Fast Fourier Transform, FFT, it seems reasonably to prefer a method where the dependency on μ is small.

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