# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Examination in: STK4060 — Time Series

Day of examination: Monday June 4'th 2012

Examination hours: 09.00 – 13.00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**Solution proposal**

## Problem 1

a) Weak stationarity: $E(X_t) = \mu$, $Var(X_t) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j^2$ independent og $t$, and autocovarinces $\gamma(h) = Cov(x_{t+h}, x_t)$ only dependent on $h$.

For all integer $h$

$$\begin{aligned}
\gamma(h) &= E[(\sum_{j=-\infty}^{\infty} \psi_j w_{t+h-j})(\sum_{j=-\infty}^{\infty} \psi_j w_{t-j})] \\
&= E[(\sum_{j=-\infty}^{\infty} \psi_{j+h} w_{t-j})(\sum_{j=-\infty}^{\infty} \psi_j w_{t-j})] \\
&= \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}
\end{aligned}$$

because $E(w_s w_t) = \sigma_w^2,\ s = t$ and $E(w_s w_t) = 0\ s \neq t$.

b) Solving backwards

$$x_t = \phi^k x_{t-k} + w_t + \cdots + \phi^k w_{t-k}.$$

Because $|\phi| < 1$,

$$E[x_t - w_t + \cdots + \phi^k w_{t-k}]^2 = \phi^{2k} E[X_{t-k}^2] \to 0$$

since $E[x_{t-k}^2] < \infty$ by assumption. Hence $x_t = \sum_{k=0}^{\infty} \phi^k w_{t-k}$, which is linear with

$$\psi_j = \begin{cases} \theta^j\ j \geq 0 \\ 0\ j < 0 \end{cases}.$$

Since $\psi_j = 0\ j < 0$, the time series is causal.

c) An ARMA(p,q) process is a weak stationary time series satisfing the stochastic difference equation

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t + \theta_1 e_{t-1} + \cdots + \theta_q w_{t-q}$$

where $w_t,\ t = 0, \pm 1, \pm 2, \ldots$ is a sequence of white noise.

The time series $x_t$ is causal if the solutiuons of $1 - \phi_1 z - \cdots - \phi_p z^p = 0$ are larger than 1 in absolute value.

The time series $x_t$ is invertible if the solutiuons of $1 + \theta_1 z - \cdots - \theta_q z^q = 0$ are larger than 1 in absolute value.

For stationary AR(p) time series the autocorrelation function (ACF), $\rho(h)$, is exponentially decreasing and the partial autocorrelation function (PACF), $\phi_{hh}$, equals zero if $h > p$.

For MA(q) time series the autocorrelation function, $\rho(h)$ equals zero if $h > q$ and the partial autocorrelation function, $\phi_{hh}$, is exponentially decreasing.

For ARMA(p,q), where $p, q > 0$ processes both ACF and PACF are decreasing.

By plotting ACF and PACF, pure AR(p) and MA(q) processes can be identified. If neither is appropriate, this suggests an ARMA(p,q).

d) For a stationary process the Yule-Walker equations are in matrix notation defined as

$$\Gamma_n \phi_n = \gamma_n \text{ and } \sigma_w^2 = \gamma(0) - \phi_n' \gamma_n$$

where $\Gamma_n$ is the covariance matrix

$$\Gamma_n = \begin{pmatrix} \gamma(0) & \cdots & \gamma(n-1) \\ \vdots & & \vdots \\ \gamma(n-1) & \cdots & \gamma(0) \end{pmatrix} \text{ and } \gamma_n = \begin{pmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{pmatrix}.$$

For an causal AR(p) time series

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t$$

one get by sucessively multiplying both sides by $x_t, x_{t-1}, \cdots, x_{t-p}$, using the assumed causality and taking expectations

$$\begin{aligned} \phi_1 \gamma(1) + \cdots + \phi_p \gamma(p) &= \gamma(0) - \sigma_w^2 \\ \phi_1 \gamma(0) + \cdots + \phi_p \gamma(p-1) &= \gamma(1) \\ \phi_1 \gamma(p-1) + \cdots \phi_p \gamma(0) &= \gamma(p) \end{aligned}$$

The first equation is $\sigma_w^2 = \gamma(0) - \phi_n' \gamma_n$ and the p last equations $\Gamma_n \phi_n = \gamma_n$.

Estimating the theoretical autocovariances, $\gamma(h)$ by the corresponding empirical, $\hat{\gamma}(h)$, one obtains estimators for the parameters $\phi_1, \cdots, \phi_p$ and $\sigma_w^2$.

e) The polynomial equation $1-(5/6)z+(1/6)z^2=0$ has solutions 3 and 2, and hence the AR(2) time series defined by

$$x_t-(5/6)x_{t-1}+(1/6)x_{t-2}=w_t$$

is causal. Using this one gets by multiplying by $x_t, x_{t-1}, \ldots$ and taking expectations

$$\begin{aligned}\gamma(0)-(5/6)\gamma(1)+(1/6)\gamma(2) &= E(x_t w_t)=\sigma_w^2\\ \gamma(1)-(5/6)\gamma(0)+(1/6)\gamma(1) &= E(x_{t-1}w_t)=0\\ \gamma(k)-(5/6)\gamma(k-1)+(1/6)\gamma(k-2) &= E(x_{t-k}w_t)=0\end{aligned}$$

for $k=2,3,\ldots$.

The covariances sayisfy the difference equation

$$\gamma(k)-(5/6)\gamma(k-1)+(1/6)\gamma(k-2)=0,\ k=1,\ldots.$$

Dividing by $\gamma(0)$ yields the diffence equation

$$\rho(k)-(5/6)\rho(k-1)+(1/6)\rho(k-2)=0,\ k=1,\ldots$$

for the autocovariance function. This difference equation has the general solution

$$\gamma(k)=c_1(1/2)^k+c_2(1/3)^k,\ k=2,\ldots$$

and initial conditions $\rho(0)=1$ and $\rho(1)-(5/6)\rho(0)+(1/6)\rho(1)=0$, i.e $\rho(0)=1$ and $\rho(1)=5/7$. Thus $c_1$ and $c_2$ are determined by

$$\begin{aligned}c_1+c_2=\rho(0)=1\\ c_1(1/2)+c_2(1/3)=\rho(1)=5/7\end{aligned}$$

which have solutions $c_1=(16/7)$ and $c_2=-(9/7)$.

Therefore the autocorrelation function is $\rho(k)=[(16/7)(1/2)^k-(9/7)(1/3)^k]/2,\ k=0,1,2,\ldots$.

The PACF is found by solving recursively the Yule-Walker equations,i.e. for h=1

$$\gamma(0)\phi_{11}=\gamma(1) \text{ or } \phi_{11}=\rho(1)=10/7.$$

Since the process is AR(2) we know that the solution $\phi_{22}=\phi_2=1/4$ and that $\phi_{kk}=0, k=3,4,\ldots$.

## Problem 2

a) The spectral density $f_x(\omega)$ is defined as

$$f_x(\omega)=\sum_{h=-\infty}^{\infty}\gamma(h)e^{-2\pi\omega h}$$

where $\gamma$ is the autocovariance function.

Large values of $f_x(\omega)$ indicates that the corresponding frequencies contribute much to the periodic variation in the time series $\{x_t\}$.

b) Using that the auto covariances are $\gamma(h) = \phi^{|h|}\sigma_w^2$ in a stationary causal AR(1) process, the spectral density can be found directly from the definition.

Alternatively, in an ARMA(p,q) process with autoregressive polynomial $\phi(z)$ and moving average polynomial $\theta(z)$ the spectral density is

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}$$

which in this case means

$$f_x(\omega) = \sigma_w^2 \frac{1}{|1 - 0.5e^{-2\pi i\omega})|^2} = \frac{\sigma_w^2}{1.25 - \cos(2\pi\omega)}.$$

c) The periodogram is defined as

$$I(j/n) = |\frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i(j/n)t}|^2, \; j = 0, 1, \ldots, n-1.$$

For $j = 1, \ldots, n-1$

$$\sum_{t=1}^{n} \mu e^{-2\pi i(j/n)t} = \mu e^{-2\pi i(j/n)} \sum_{t=0}^{n-1} (e^{-2\pi i(j/n)})^t = \mu e^{-2\pi i(j/n)} \frac{1 - e^{-2\pi i(j/n)n}}{1 - e^{-2\pi i(j/n)}} = 0.$$

Therefore, for $j = 1, \ldots, n-1$

$$I(j/n) = |\frac{1}{\sqrt{n}} \sum_{t=1}^{n} (x_t - \mu) e^{-2\pi i(j/n)t}|^2 = \frac{1}{n} \sum_{s,t=1}^{n} (x_s - \mu)(x_t - \mu)(e^{-2\pi i(j/n)(s-t)}.$$

Hence,

$$E[I(j/n)] = \frac{1}{n} \sum_{s,t=1}^{n} \gamma(s-t) e^{-2\pi i(j/n)(s-t)} = \frac{1}{n} \sum_{h=-(n-1)}^{n-1} (n-|h|)\gamma(h) e^{-2\pi i(j/n)h}.$$

by changing summation variable $h = s - t$.

d) For $j \neq k$ $I(j/n)$ and $I(k/n)$ are for large values of $n$ under the regularity conditions described in Shumway and Stoffer, in particular $\sum_{h=-\infty}^{\infty} |h|\gamma(h) < \infty$, approximately uncorrelated. For linear processes $(\frac{2I(j_1/n)}{f_x(j_1/n)}, \ldots, \frac{2I(j_k/n)}{f_x(j_k/n)})'$ is moreover approximately distributed as $(X_1, \ldots, X_k)'$ where $X_1, \ldots, X_k$ are independently $\chi^2$-distributed with 2 degrees of fredom.

The spectral density $f_x$ is continous and if it is not varying too much, gain can be obtained by smoothing/averaging over adjacent frequencies. But there is a tradeoff, if the smooting is too coarse, bias will result when $f_x$ is not constant, and if not enough frequencies are included in the smooth the variance will increase.

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