Term paper (STK 4060)

April 21, 2008

Deadline for handing in: May 2, 2008.
The paper must be delivered at Ekspedisjonen, 7th floor, N.H.Abels house (before 3 p.m).

Problem 1

a) Define the spectral density $f_X(\lambda)$ of a zero-mean stationary time series X_t. Try to interpret $f_X(\lambda)$ in the case where $f_X(\lambda)$ has a single peak at the frequency $\lambda = 2\pi/s$ for some positive integer s.

b) Assume that $X_t = Z_t + \theta Z_{t-1}$, with $Z_t \sim WN(0, \sigma^2)$. Derive an expression for $f_X(\lambda)$ in this case.

Now consider the AR(3) model

$$\phi(B)X_t = Z_t, \tag{1}$$

where B is the backshift operator: $B^k X_t \equiv X_{t-k}$, and

$$\phi(B) = (1 + r^{-1}B)(1 - r^{-1}e^{i\frac{\pi}{2}}B)(1 - r^{-1}e^{-i\frac{\pi}{2}}B),\ r > 0 \tag{2}$$

c) Show that (1)-(2) is equivalent to

$$X_t + \phi X_{t-1} + \phi^2 X_{t-2} + \phi^3 X_{t-3} = Z_t$$

for a suitable choice of parameter ϕ. What conditions must be satisfied in order for X_t to be i) stationary and ii) causal?

In the rest of the problem, assume that $r > 1$ and $\sigma^2 = 1$.

d) Derive the spectral density $f_X(\lambda)$ of the AR(3) process (1)-(2). Plot the density over $\lambda \in [0, \pi]$ for $r = 2$, 1.5 and 1.01, respectively. Give an interpretation of the density when $r = 1.01$. In this case, what does the spectral density say about the behavior of X_t?

e) Find the difference equations that determine the coefficients ψ_i in the MA-representation $X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i}$, and find these coefficients when $r = 1.01$. Use the coefficients to calculate and plot the autocovariance function $\gamma_X(h)$. Does it appear that X_t has a particular cyclical pattern? How does the pattern of $\gamma_X(h)$ correspond to that of $f_X(\lambda)$?

Now consider the transformation $Y_t = (1 - B^4)X_t$.

f) Show that Y_t is an ARMA(3,4) process.

g) Find the roots of $1 - z^4$, i.e. the solutions of $1 - z^4 = 0$. Derive the spectral density of Y_t, i.e. $f_Y(\lambda)$, and plot this density over $\lambda \in [0, \pi]$ when $r = 1.01$. Interpret the results! Does it appear that Y_t has a particular cyclical pattern?

h) Now let $r \to 1$ (from above), to derive the spectral density of Y_t when $r = 1$. Show that this limit is well defined for all λ and, indeed, is the spectral density of an MA(q) process

$$Y_t = \theta^*(B)Z_t. \tag{3}$$

What is $\theta^*(B)$?

i) Based on (3), derive an expression for the one- and four-step linear predictors $P_n X_{n+1}$ and $P_n X_{n+4}$ and the corresponding mean squared prediction errors.

Problem 2

In this problem you shall use the data in the accompanying data file QGDP.txt. The file contains observations on quarterly Gross Domestic Product (GDP) for Norway for the period 1979Q4 to 2007Q2.

a) Generate a new time series, $\{X_t\}_{t=1}^{n}$ by taking the (natural) logarithm of the original variables, but exclude the last 8 observations for prediction purposes – see f) below. Plot the resulting series. Does the series appears to be stationary?

b) Generate the differenced series $Y_t = X_t - X_{t-1}$ and plot Y_t. Calculate the (empirical) autocorrelation and partial autocorrelation functions for Y_t. Comment!

c) Estimate $\mu_Y = E(Y_t)$ and give an approximate 95% confidence interval for μ_Y. What is the interpretation of μ_Y in terms of GDP? State explicitly the assumptions you make for valid inference and discuss in view of b). Subtract the mean, $\overline{Y}$, from Y_t and plot the periodogram based on this data. Estimate the spectral density of

$Y_t^* = Y_t - \mu_Y$ (assuming that this is well-defined) by smoothing the periodogram, as explained in Section 4.2 in Brockwell and Davis. Comment on the results.

d) Find an appropriate seasonal ARIMA (SARIMA) model to represent the series X_t obtained in a). Explain and justify your choice of model and give approximate confidence intervals for estimated coefficients based on the method of maximum likelihood.

e) Plot the standardized residuals from the fitted model – and explain how you have obtained these. Evaluate the quality of your chosen model from the the properties of the residuals, e.g. by examining their autocorrelation function and performing goodness-of-fit tests.

f) Use the fitted model to predict values for the 8 excluded observations, i.e. calculate $P_n X_{n+1}$,..., $P_n X_{n+8}$. Also obtain 95% confidence intervals for each predicted value. Plot and compare the predicted and actual values together with the confidence intervals.

g) Whatever your chosen specification in d), what is the implied distribution of $Y_t = (1 - B)X_t$? Is Y_t a stationary process according to your estimated model?