Problem 1

Let $x_1, \ldots, x_n$ be observations from a weakly stationary time series where $E[x_t] = \mu$ and the auto covariances satisfy $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. Denote the spectral density by $f_x(\omega), \omega \in [-\frac{1}{2}, \frac{1}{2}]$.

a) Let $\omega_j = j/n, j = 0, 1, \ldots, n-1$ denote the fundamental frequencies and let $I(\omega_j), j = 0, 1, \ldots, n-1$ be the periodogram. Show that the periodogram has expectation

$$E[I(\omega_j)] = \sum_{h=-(n-1)}^{(n-1)} (1 - \frac{|h|}{n})\gamma(h)e^{-2\pi i\omega_j h}, j = 1, \ldots, n-1$$

and that when $j = 0$

$$E[I(0)] - n\mu^2 = \sum_{h=-(n-1)}^{(n-1)} (1 - \frac{|h|}{n})\gamma(h).$$

b) Show that the spectral density $f_x(\omega)$ is continous under the conditions stated above. Use this to prove that if $\{j_n\}$ is a sequence such that $j_n/n \to \omega \neq 0$ as $n \to \omega$, then $E[I(\omega_{j_n})] \to f_x(\omega)$.

c) This part is a part of problem 4.15 in the textbook by Shumway and Stoffer, i.e. if $h_t, t = 1, \cdots, n$ are numbers and $y_t = h_t x_t$, show that then $\{y_t\}$ has periodogram with expectation

$$E[I_y(\omega_j)] = \int_{-\frac{1}{2}}^{\frac{1}{2}} |H_n(\omega_j - \omega)|^2 f_x(\omega) d\omega.$$

where $H_n(\omega) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} h_t e^{-2\pi i\omega t}$.