Problem 3.2

a) $x_t = \phi x_{t-1} + w_t, t = 1, 2, \ldots, x_0 = w_0$.
Substituting recursively

$$\begin{aligned} x_t = & \phi x_{t-1} + w_t, = \phi(\phi x_{t-1} + w_{t-1}) + w_t \\ = & \phi^k x_{t-k} + \Sigma_{j=0}^{k-1} \phi^j w_{t-j} \\ = & \phi^{t-1} x_1 + \Sigma_{j=0}^{t-2} \phi^j w_{t-j} \\ = & \phi^{t-1}(\phi x_0 + w_0) + \Sigma_{j=0}^{t-2} \phi^j w_{t-j} \\ = & \phi^{t-1}(\phi w_0 + w_0) + \Sigma_{j=0}^{t-2} \phi^j w_{t-j} \\ = & \Sigma_{j=0}^{t} \phi^j w_{t-j} \end{aligned}$$

b) $E(w_t) = 0$ so $E(x_t) = 0$.

c) Because $w_0, \ldots, w_t$ are uncorrelated

$$var(x_t) = var(\Sigma_{j=0}^{t} \phi^j w_{t-j}) = \Sigma_{j=0}^{t} \phi^{2j} \sigma_w^2 = \frac{1 - \phi^{2t+2}}{1 - \phi^2} \sigma_w^2.$$

d) From part a) $x_t = \phi^k x_{t-k} + \Sigma_{j=0}^{k-1} \phi^j w_{t-j}$. Evaluate at $t+h$ and let $h = k$ so

$$x_{t+h} = \phi^k x_t + \Sigma_{j=0}^{h-1} \phi^j w_{t+h-j}.$$

Then $E(x_{t+h} x_t) = E(\phi^h x_t^2) + E[(w_{t+h} + \cdots + \phi^{h-1} w_{t+1}) x_t] = \phi^h var(x_t)$.

e) $var(x_t)$ is not constant in t, so $\{x\}$ cannot be stationary.

f) $x_t = \Sigma_{j=0}^{t} \phi^j w_{t-j}$ converges in mean square toward $x_t = \Sigma_{j=0}^{\infty} \phi^j w_{t-j}$ when $t \to \infty$, which is stationary.

g) Simulate first a "burn-in" period using $w_t \sim iidN(0, \sigma^2)$. For a long enough period the last variable is approximately stationary and can be used for generating n additional variables using the autoregressive recursion.

h) Can use induction.
First let t=1. Then $var(x_1) = var(\phi x_0 + w_1) = \phi^2 \frac{\sigma^2}{1-\sigma^2} + \sigma^2 = \sigma^2 \frac{1}{1-\sigma^2}$.
Next, assume $x_t = \sigma^2 \frac{1}{1-\sigma^2}$. Then $var(x_{t+1}) = var(\phi x_t + w_{t+1}) = \phi^2 \frac{\sigma^2}{1-\sigma^2} + \sigma^2 = \sigma^2 \frac{1}{1-\sigma^2}$.

Therefore $var(x_{t+1}) = \sigma^2 \frac{1}{1-\sigma^2}$. Using, from part a) $x_t = \phi^h x_{t-h} + \Sigma_{j=0}^{h-1} \phi^j w_{t-j}$

$$E(x_{t+h}x_t) = \phi^{2h} var(x_t) + E[(w_{t+h} + \cdots + \phi^{h-1} w_{t+1})x_t] = \frac{\phi^{2h}\sigma^2}{1-\sigma^2}.$$

which only depend on h.