Problem 3.5

Proposition 1 *Consider the AR(2)-model* $(1 - \phi_1 B - \phi_2 B^2)x_t = w_t$ *where* $w_t \sim wn(0, \sigma_w^2)$. *Then the following two statements are equivalent.*

(i) $\phi(z) = (1 - \phi_1 z - \phi_2 z^2) = 0 \Rightarrow |z| > 1$

(ii) $\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1$ *and* $|\phi_2| < 1$.

Proof. $(i) \Rightarrow (ii)$: As in class
$(ii) \Rightarrow (i)$: Note the following:

$$\begin{aligned} \phi(1) &= 1 - \phi_1 - \phi_2 > 0, \\ \phi(-1) &= 1 + \phi_1 - \phi_2 > 0, \\ \phi(0) &= 1 \end{aligned} \tag{1}$$

and

$$\phi'(z) = -\phi_1 - 2\phi_2 z, \quad \phi'(1) = -\phi_1 - 2\phi_2, \quad \phi'(-1) = -\phi_1 + 2\phi_2 \tag{2}$$

For $\phi_2 = 0$ the result is true so it suffices to consider the following cases:

a) Two real roots of $\phi(z) = 0, \phi_2 > 0$.
Since $\phi(z)$ is a parabola and $\phi(z) \to -\infty$ as $z \to \pm\infty$ and $\phi(-1), \phi(0), \phi(1) > 0$ by (1), the solutions of $\phi(z) = 0$ must be outside $[-1, 1]$.

b) Two real roots of $\phi(z) = 0, \phi_2 < 0, \phi_1 > 0$.
By (1) $\phi(0), \phi(1) > 0$. Since the roots are real, $\phi_1^2 + 4\phi_2 > 0$. From (2) it follows that $\phi'(0) = -\phi_1 < 0$ and $\phi'(1) = -\phi_1 - 2\phi_2 < -\phi_1 + \phi_1^2/2 < 0$ becauce $0 < \phi_1 < 2$. Using that $\phi(z)$ is a parabola and $\phi(z) \to \infty$ $z \to \pm\infty$ it follows that $\phi(z)$ must be decreasing in $[0, 1]$ so the roots of $\phi(z) = 0$ are larger than 1.

c) Two real roots of $\phi(z) = 0, \phi_2 < 0, \phi_1 < 0$.
By (1) $\phi(-1), \phi(0) > 0$. Now $\phi'(-1) = -\phi_1 + 2\phi_2 > -\phi_1 - \phi_1^2/2 > 0$ because $-2 < \phi_1 < 0$. Hence arguing as in b) $\phi(z)$ must be increasing in $[-1, 0]$ so the roots of $\phi(z) = 0$ are smaller than -1.

d) Two complex roots of $\phi(z) = 0$.
Since z_1 and z_2 are the roots, $z_2 = \bar{z}_1$ and $\phi(z) = (1 - \frac{1}{z_1}z)(1 - \frac{1}{\bar{z}_1}z) = 1 - \phi_1 z - \phi_2 z^2$. Hence $\frac{1}{|z_1|^2} = |\phi_2| < 1$ and $|z_1| = |z_2| > 1$.